Fourier Transforms and potentials

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Contents

1 Potentials

Let $V(r)$ be a spherically symmetric 3-dimensional potential

$$V(\vec{r}) = V(r)$$

with $r = \|\vec{r}\|$.

Let's suppose the potential has the form

$$V_\alpha(r) = \kappa r^\alpha$$

Since the potential depends explicitly only on $r = \|\vec{r}\|$, The ϕ and θ azimuthal and polar angle can be done explicitly. The Fourier transform then depends only on the radial component of the wave vector $\tilde{V} = V(q)$ with $q = \|\vec{q}\|$.

1.1 Fourier transform

The Fourier transform of V is then

$$\tilde{V}_\alpha(q) = \int d^3r V_\alpha(r) e^{i\vec{q}\cdot\vec{r}} = 2\pi \int_0^\infty r^2 V(r)\, dr \int_{-1}^1 e^{iqrz} dz$$

where $z = \cos(\theta)$ with $\theta \in [0,\pi]$ and the integration on $\phi \in [0, 2\pi]$ is already performed. We used the volume element $d^3r = rd\theta\, r\sin(\theta) d\phi\, dr = r^2 d\big(\cos(\theta)\big) d\phi\, dr$.

Since by following θ, $z = \cos(\theta)$ goes from 1 to -1, reversing the limits of integration cancels the minus sign from the differential of z.

Doing the z integration we are left with

$$\tilde{V}_\alpha(q) = 2\pi \int_0^\infty r^2 V_\alpha(r) \frac{e^{iqr} - e^{-iqr}}{iqr} dr = \frac{4\pi}{q} \int_0^\infty r V_\alpha(r) \sin(qr) dr$$

For the central potential mentioned above, we have then

$$\tilde{V}_\alpha(r) = \frac{4\pi}{q} \kappa \int_0^\infty r^{\alpha+1} \sin(qr) dr$$

1.2 Dirac distribution

The functional form of the Dirac distribution is defined by

$$f(x) = \int f(y) \delta(x - y) dy$$

So the Fourier transform of δ is

$$\tilde{\delta}(q) = \int \delta(x) e^{iqx} = e^{iq0} = 1.$$

1.3 Inverse Fourier transform and Dirac distribution

The inverse Fourier transform of a function is defined as

$$f(x) = \frac{1}{2\pi} \int \tilde{f}(q) e^{-iqx}.$$

If we apply this definition to the Fourier transform of the Dirac function, we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iqx} dq$$

which does not converge. To make it convergent we can add a term $-\epsilon q$ in the exponential

$$\frac{1}{\pi} \int e^{-iqx - \epsilon|q|} dq$$

and make $\epsilon \to 0$ at the end of the integration. We get then

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\left[\frac{e^{-iqx-\epsilon q}}{-ix-\epsilon}\right]_0^\infty + \frac{1}{2\pi}\left[\frac{e^{-iqx+\epsilon q}}{-ix+\epsilon}\right]_{-\infty}^0 \\
&= \frac{1}{2\pi}\left(\frac{1}{ix+\epsilon} + \frac{1}{\epsilon - ix}\right) \\
&= \frac{1}{2\pi}\frac{2\epsilon}{\epsilon^2+x^2} \\
&= \frac{1}{\pi}\frac{\epsilon}{\epsilon^2+x^2}.
\end{aligned}
$$

As $x \neq 0$, in the limit $\epsilon \to 0$ we have $f(x) = 0$.

Before taking the limit $\epsilon \to 0$ when $x = 0$, let's calculate

$$
\int_{-\infty}^{\infty} f(x)dx = \frac{\epsilon}{\pi}\int \frac{dx}{\epsilon^2+x^2} = \frac{\epsilon}{\pi}\frac{1}{\epsilon}\arctan(\frac{x}{\epsilon})\Big|_{-\infty}^{\infty} = 1.
$$

So that the integral is independent of ϵ. Moreover, when $x = 0$ we have

$$
f(0) = \lim_{\epsilon\to 0}\frac{1}{\epsilon\pi} = \infty
$$

This means that f is everywhere nul except in zero where it diverges and its integral is unity. This looks a lot like the Dirac distribution functional form.

So, in all generality, we can state that

$$
f(x) = \frac{1}{\pi}\lim_{\epsilon\to 0}\frac{\epsilon}{\epsilon^2+x^2} = \delta(x).
$$

1.4 Inverse Fourier transform

The inverse Fourier transform of a Fourier transform $\tilde{V}$ is given by

$$
V(r) = \frac{1}{(2\pi)^3}\int d^3q\, V(q)e^{-i\vec{q}\cdot\vec{r}}.
$$

If $\tilde{V}(q)$ is the Fourier tranform of $V(r)$

$$
\tilde{V}(q) = \int d^3rV(r)e^{i\dot{\vec{q}\vec{r}}}
$$

then the inverse Fourier transform of $\tilde{V}$ is

$$
\begin{aligned}
\frac{1}{(2\pi)^3}\int d^3q\int d^3rV(r)e^{i\vec{q}\cdot\vec{r}-i\vec{q}\cdot\vec{r}'} &= \frac{1}{(2\pi)^3}\int d^3rV(r)\int d^3qe^{i\vec{q}\cdot(\vec{r}-\vec{r}')} \\
&= \int d^3rV(r)\delta^3(\vec{r}-\vec{r}') \\
&= V(r')
\end{aligned}
$$

So, as its name suggests, the inverse Fourier transform gives back the original function.

2 Coulomb-like potential

In the cas of the Coulomb-like potential we have $\alpha = -1$

$$V_{-1}(r) = \kappa \frac{1}{r}$$

and the Fourier transform of V_{-1} is

$$\tilde{V}_{-1}(q) = \frac{4\pi}{2iq}\kappa \int_0^\infty \left(e^{(iq-\epsilon)r} - e^{-(iq+\epsilon)r} \right) dr$$

where ϵ is to be taken as $\epsilon \to 0$ at the end of the integration, to make the integral finite. So we get

$$\begin{aligned}
\tilde{V}_{-1}(q) &= \frac{4\pi\kappa}{2iq} \left[\frac{e^{(iq-\epsilon)r}}{iq-\epsilon} + \frac{e^{-(iq+\epsilon)r}}{iq+\epsilon} \right]_0^\infty \\
&= \frac{4\pi\kappa}{2iq} (- \frac{1}{iq} - \frac{1}{iq}) \\
&= -\frac{4\pi\kappa}{2iq} \frac{2}{iq}
\end{aligned}$$

That is

$$\boxed{\tilde{V}_{-1}(q) = \frac{4\pi\kappa}{q^2}.}$$

Clearly, we have also

$$\boxed{V_{-1}(r) = \frac{1}{(2\pi)^3} \int d^3q \frac{4\pi\kappa}{q^2} e^{-i\vec{q}\cdot\vec{r}} = \frac{\kappa}{r}.}$$

3 Constant potential

The constant potential case is given by $\alpha = 0$

$$V(r) = \kappa.$$

Here we have

$$\begin{aligned}
\tilde{V}_0(q) &= \frac{4\pi}{2iq}\kappa \int_0^\infty r \left(e^{(iq-\epsilon)r} - e^{-(iq+\epsilon)r} \right) dr \\
&= \frac{4\pi}{2iq}\kappa \int_0^\infty \frac{1}{i}\frac{\partial}{\partial q} \left(e^{(iq-\epsilon)r} + e^{-(iq+\epsilon)r} \right) dr \\
&= \frac{4\pi}{2iq}\kappa \frac{1}{i}\frac{\partial}{\partial q} \left[\frac{e^{(iq-\epsilon)r}}{iq-\epsilon} - \frac{e^{-(iq+\epsilon)r}}{iq+\epsilon} \right]_0^\infty
\end{aligned}$$

That is

$$\begin{aligned}\tilde{V}_0(q) &= \frac{4\pi}{2iq}\kappa\frac{1}{i}\frac{\partial}{\partial q}\left[-\frac{1}{iq-\epsilon}+\frac{1}{iq+\epsilon}\right]\\ &= \frac{4\pi}{2iq}\kappa\frac{1}{i}\frac{\partial}{\partial q}\frac{-2\epsilon}{q^2+\epsilon^2}\end{aligned}$$

and with $\epsilon \to 0$ we recover

$$\boxed{\tilde{V}_0(q) = \frac{4\pi}{q}\kappa\delta'(q)}$$

where $\delta'(q) = \frac{\partial}{\partial q}\delta(q)$.

4 Linear potential

For $\alpha = 1$ we have

$$\begin{aligned}\tilde{V}_1(q) &= \frac{4\pi}{2iq}\kappa\int_0^\infty r^2\left(e^{(iq-\epsilon)r} - e^{-(iq+\epsilon)r}\right)dr\\ &= \frac{4\pi}{2iq}\kappa\int_0^\infty \frac{1}{i^2}\frac{\partial^2}{\partial q^2}\left(e^{(iq-\epsilon)r} - e^{-(iq+\epsilon)r}\right)dr\\ &= \frac{4\pi}{2iq}\kappa\frac{\partial^2}{\partial q^2}\left[\frac{1}{iq-\epsilon}+\frac{1}{iq+\epsilon}\right]_0^\infty\\ &= -\frac{4\pi}{2iq}\kappa\frac{\partial^2}{\partial q^2}\frac{2iq}{q^2+\epsilon^2}\\ &= -\frac{4\pi}{q}\kappa\frac{\partial^2}{\partial q^2}\frac{1}{q}\end{aligned}$$

where we let $\epsilon \to 0$ in the last step.

So we finally get

$$\boxed{\tilde{V}_1(q) = -\frac{8\pi}{q^4}\kappa.}$$

5 Yukawa potential

Let us consider

$$V_Y(r) = \frac{e^{-\alpha r}}{r}$$

We have then, where the ϵ converging factor has been omitted, since the exponential with α makes the integral converge

$$\begin{aligned}
\tilde{V}_Y(q) &= \frac{4\pi}{2iq} \int_0^\infty \left(e^{(iq-\alpha)r} - e^{-(iq+\alpha)r} \right) dr \\
&= \frac{4\pi}{2iq} \left[\frac{e^{(iq-\alpha)r}}{iq-\alpha} + \frac{e^{-(iq+\alpha)}}{iq+\alpha} \right]_0^\infty \\
&= -\frac{4\pi}{2iq} \left(\frac{1}{iq-\alpha} + \frac{1}{iq+\alpha} \right) \\
&= \frac{4\pi}{2iq} \frac{2iq}{q^2+\alpha^2}
\end{aligned}$$

So that

$$\boxed{\tilde{V}_Y(q) = \frac{4\pi}{q^2+\alpha^2}.}$$

6 Yukawa-like potentials

We can now consider potentials of the form

$$V_{Y\alpha} = r^\alpha e^{-\alpha r}$$

for which the Fourier transform is given by

$$\tilde{V}_{Y\alpha}(q) = \frac{4\pi}{2iq} \int_0^\infty r^{\alpha+1} \left(e^{(iq-\alpha)r} - e^{-(iq+\alpha)r} \right) dr$$

These are the same integrals found before except that now, we do not take $\alpha \to 0$.