Exercises from Feynman and Hibbs

Quantum Mechanics and Path Integrals

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The Fundamental Concepts of Quantum Mechanics

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The Quantum-mechanical Law of Motion

Exercise 2.1

For a free particle $L = \frac{1}{2}m\dot{x}^2$. We have

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial L}{\partial x} = 0.$$

The Euler-Lagrange equation is then

$$m\ddot{x} = 0$$

so that

$$\dot{x} - \dot{x}_a = v \quad \text{and} \quad x - x_a = (v + v_a)(t - t_a).$$

so that, with $v_a = \dot{x}_a$ and $V = v + v_a$

$$x = x_a + V(t - t_a)$$

With the conditions at t_b

$$x_b = x_a + V(t_b - t_a) \qquad \Rightarrow \qquad V = \frac{x_b - x_a}{t_b - t_a}.$$

Finally

$$x = x_a + \frac{x_b - x_a}{t_b - t_a}(t - t_a) \qquad \text{and} \qquad \dot{x} = \frac{x_b - x_a}{t_b - t_a}.$$

The Lagrangian for the classical path is then

$$L = \frac{m}{2}\left(\frac{x_b - x_a}{t_b - t_a}\right)^2$$

a constant.

The classical action is then the integral

$$S_{\text{cl}} = \int_{t_a}^{t_b} dt\, L = \frac{m}{2}\left(\frac{x_b - x_a}{t_b - t_a}\right)^2 (t_b - t_a).$$

That is

$$\boxed{S_{\text{cl}} = \frac{m}{2}\frac{(x_b - x_a)^2}{t_b - t_a}}$$

Exercise 2.2

For a harmonic oscillator $L = \frac{m}{2}(\dot{x}^2 - \omega^2 x^2)$.

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial L}{\partial x} = -m\omega^2 x.$$

and the Euler-Lagrange equation reads

$$\ddot{x} + \omega^2 x = 0.$$

To integrate the equation perform a change in variable

$$z = z(x) = \dot{x}, \qquad \text{so that} \qquad \ddot{x} = z'(x)\dot{x} = zz'$$

The differential equation now reads

$$zz' = -\omega^2 x$$

and we can integrate

$$\int_{z_a}^{z} dz\, zz' = -\omega^2 \int_{x_a}^{x} dx\, x$$

to obtain

$$\frac{1}{2}(z^2 - z_a^2) = -\frac{1}{2}\omega^2(x^2 - x_a^2)$$

We can then write

$$z^2 = z_a^2 + \omega^2 x_a^2 - \omega^2 x^2$$

and restore $\dot{x}$ instead of z

$$\dot{x} = \sqrt{z_a^2 + \omega^2 x_a^2 - \omega^2 x^2}$$

and integrate again

$$\int_{x_a}^{x} \frac{dx}{\sqrt{1 - \left(\frac{\omega x}{\zeta}\right)}} = \int_{t_a}^{t} \zeta\, dt$$

where $\zeta^2 = z_a^2 + \omega^2 x_a^2$. The integration gives

$$\frac{\zeta}{\omega}\left(\arcsin(\frac{\omega x}{\zeta}) - \arcsin(\frac{\omega x_a}{\zeta})\right) = \zeta(t - t_a)$$

that is

$$\arcsin(\frac{\omega x}{\zeta}) = \arcsin(\frac{\omega x_a}{\zeta}) + \omega(t - t_a)$$

and taking the sine, and using the formula $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

$$\frac{\omega x}{\zeta} = \frac{\omega x_a}{\zeta}\cos(\omega(t - t_a)) + \cos(\arcsin(\frac{\omega x_a}{\zeta}))\sin(\omega(t - t_a))$$

and the last $\cos(\arcsin(\dots)) = \sqrt{1 - (\omega^2 x_a^2/\zeta^2)} = \frac{z_a}{\zeta}$

$$\frac{\omega x}{\zeta} = \frac{\omega x_a}{\zeta}\cos(\omega(t - t_a)) + \frac{z_a}{\zeta}\sin(\omega(t - t_a))$$

We then get

$$x = x_a \cos(\omega(t - t_a)) + \frac{z_a}{\omega} \sin(\omega(t - t_a))$$

Replacing t with t_b and x with x_b we can express z_a

$$z_a = \frac{x_b - x_a \cos(\omega T)}{\sin(\omega T)} \omega$$

where $T = t_b - t_a$. So we have

$$x = x_a \cos(\omega(t - t_a)) + \frac{x_b - x_a \cos(\omega T)}{\sin(\omega T)} \sin(\omega(t - t_a))$$

and

$$\dot{x} = \omega \left(-x_a \sin(\omega(t - t_a)) + \frac{x_b - x_a \cos(\omega T)}{\sin(\omega T)} \cos(\omega(t - t_a)) \right)$$

Let $\ell = \frac{2L}{m\omega^2} = \frac{\dot{x}}{\omega^2} - x^2)$, and $\tau = t - t_a$, we have

$$\dot{x}^2 = \omega^2 \left(x_a^2 \sin^2(\omega\tau) + \frac{(x_b - x_a \cos(\omega T))^2}{\sin^2(\omega T)} \cos^2(\omega\tau) - \frac{x_a(x_b - x_a \cos(\omega T))}{\sin(\omega T)} \sin(2\omega\tau) \right)$$

and

$$x^2 = x_a^2 \cos^2(\omega\tau) + \frac{(x_b - x_a \cos(\omega T))^2}{\sin^2(\omega T)} \sin^2(\omega\tau) + \frac{x_a(x_b - x_a \cos(\omega T))}{\sin(\omega T)} \sin(2\omega\tau)$$

so that

$$\begin{aligned}
\ell &= \frac{\dot{x}}{\omega^2} - x^2 \\
&= \cos(2\omega\tau) \left(-x_a^2 + \frac{(x_b - x_a \cos(\omega T))^2}{\sin^2(\omega T)} \right) - 2\frac{x_a(x_b - x_a \cos(\omega T)}{\sin(\omega T)} \sin(2\omega\tau) \\
&= \cos(2\omega\tau) \left(\frac{-x_a^2 \sin^2(\omega T) + x_b^2 + x_a^2 \cos^2(\omega T) - 2x_a x_b \cos(\omega T)}{\sin^2(\omega T)} \right) - 2\frac{x_a x_b - x_a^2 \cos(\omega T)}{\sin(\omega T)} \sin(2\omega\tau) \\
&= \cos(2\omega\tau) \left(\frac{x_a^2 \big(1 - 2\sin^2(\omega T)\big) + x_b^2 - 2x_a x_b \cos(\omega T)}{\sin^2(\omega T)} \right) - 2\frac{x_a x_b - x_a^2 \cos(\omega T)}{\sin(\omega T)} \sin(2\omega\tau) \\
&= \cos(2\omega\tau) \left(\frac{x_a^2 \cos(2\omega T) + x_b^2 - 2x_a x_b \cos(\omega T)}{\sin^2(\omega T)} \right) - 2\frac{x_a x_b - x_a^2 \cos(\omega T)}{\sin(\omega T)} \sin(2\omega\tau)
\end{aligned}$$

On integrating between t_a and t_b, we get

$$\begin{aligned}
\frac{2}{m\omega^2} S_{\text{cl}} &= \int_{t_a}^{t_b} dt\, \cos(2\omega\tau) \left(\frac{x_a^2 \cos(2\omega T) + x_b^2 - 2x_a x_b \cos(\omega T)}{\sin^2(\omega T)} \right) - 2\frac{x_a x_b - x_a^2 \cos(\omega T)}{\sin(\omega T)} \sin(2\omega\tau) \\
&= \frac{1}{2\omega} \Big(\sin(2\omega T) - \sin(0) \Big) \left(\frac{x_a^2 \cos(2\omega T) + x_b^2 - 2x_a x_b \cos(\omega T)}{\sin^2(\omega T)} \right) \\
&\qquad + 2\frac{x_a x_b - x_a^2 \cos(\omega T)}{\sin(\omega T)} \Big(\cos(2\omega T) - \cos(0) \Big) \frac{1}{2\omega} \\
&= \frac{1}{2\omega} \sin(2\omega T) \left(\frac{x_a^2 \cos(2\omega T) + x_b^2 - 2x_a x_b \cos(\omega T)}{\sin^2(\omega T)} \right) + 2\frac{x_a x_b - x_a^2 \cos(\omega T)}{\sin(\omega T)} (\cos(2\omega T) - 1) \frac{1}{2\omega}
\end{aligned}$$

So, with $\sin(\omega T) = 2\sin(\omega T)\cos(\omega T)$

$$S_{\text{cl}} = \frac{m\omega}{2}\left[\cos(\omega T)\left(\frac{x_a^2\cos(2\omega T) + x_b^2 - 2x_a x_b\cos(\omega T)}{\sin(\omega T)}\right) + \frac{x_a x_b - x_a^2\cos(\omega T)}{\sin(\omega T)}(\cos(2\omega T) - 1)\right]$$
$$= \frac{m\omega}{2\sin(\omega T)}\left[\cos(\omega T)x_a^2\cos(2\omega T) + x_b^2\cos(\omega T) - 2x_a x_b\cos^2(\omega T) + (x_a x_b - x_a^2\cos(\omega T))(\cos(2\omega T) - 1)\right]$$

$$S_{\text{cl}} = \frac{m\omega}{2\sin(\omega T)}\Big[x_a^2\cos(2\omega T)\cos(\omega T) + x_b^2\cos(\omega T) - 2x_a x_b\cos^2(\omega T) + (x_a x_b - x_a^2\cos(\omega T)))\cos(2\omega T) - (x_a x_b - x_a^2\cos(\omega T)))\Big]$$

Thus

$$\boxed{S_{\text{cl}} = \frac{m\omega}{2\sin(\omega T)}\left((x_a^2 + x_b^2)\cos(\omega T) - 2x_a x_b\right)}$$

Exercise 2.3

Find S_{cl} for a particle under a constant force F, that is, $L = \frac{m}{2}\dot{x}^2 - Fx$.

We have

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{partial L}{\partial x} = -F$$

so that the Euler-Lagrange equation reads

$$m\ddot{x} + F = 0$$

from which we have after a first integration between t_a and t_b

$$\dot{x} - \dot{x}_a = -\frac{F}{m}(t - t_a)$$

with $\alpha = \frac{F}{m}$ and $\tau = t - t_a$ we have then

$$\dot{x} = \dot{x}_a - \alpha\tau$$

On a second integration we have

$$x - x_a = \dot{x}_a\tau - \frac{1}{2}\alpha\tau^2$$

When $t = t_b$, that is $\tau = T = t_b - t_a$, we want $x = x_b$

$$x_b - x_a = \dot{x}_a T - \frac{1}{2}\alpha T^2$$

so that we can express $\dot{x}_a$

$$\dot{x}_a = \frac{x_b - x_a}{T} + \frac{1}{2}\alpha T$$

Thus

$$x = x_a + \left(\frac{x_b - x_a}{T} + \frac{1}{2}\alpha T\right)\tau - \frac{1}{2}\alpha\tau^2$$

and

$$\dot{x} = \frac{x_b - x_a}{T} + \frac{1}{2}\alpha T - \alpha\tau$$

For the classical path, the lagrangian is

$$L = \frac{1}{2}m\left(\frac{x_b - x_a}{T} + \frac{1}{2}\alpha T - \alpha\tau\right)^2 - m\alpha\left[x_a + \left(\frac{x_b - x_a}{T} + \frac{1}{2}\alpha T\right)\tau - \frac{1}{2}\alpha\tau^2\right]$$

After expanding the square

$$L = \frac{1}{2}m\left(\frac{(x_b - x_a)^2}{T^2} + \frac{1}{4}\alpha^2T^2 + \alpha^2\tau^2 + \alpha(x_b - x_a) - 2\alpha\frac{x_b - x_a}{T}\tau - \alpha^2T\tau\right)$$
$$- m\alpha x_a - m\alpha\frac{x_b - x_a}{T}\tau - \frac{1}{2}m\alpha^2T\tau + \frac{1}{2}m\alpha^2\tau^2$$

and simplifying we get

$$L = \frac{1}{2}m\left(\frac{x_b - x_a}{T}\right)^2$$

$$\boxed{S_{\text{cl}} = -\frac{F^2}{24m}(t_b - t_a)^3 - \frac{F}{2}(x_b + x_a)(t_b - t_a) + \frac{m}{2}\frac{(x_b - x_a)^2}{t_b - t_a}}$$

Exercise 2.4

Classically, the momentum is defined s $p = \frac{\partial L}{\partial \dot{x}}$. Show that the momentum at an end point is

$$\left(\frac{\partial L}{\partial \dot{x}}\right)_{x = x_b} = \frac{\partial S_{\text{cl}}}{\partial x_b}$$

Exercise 2.5

Classically the energy is defined as

$$E = \dot{x}p - L$$

Show that at an end point

$$E_b = \frac{\partial S_{\text{cl}}}{\partial t_b}.$$

Exercise 2.6

Consider a one-dimensional relativistic particle that can move forward and backward at the velocity of light. We consider that the velocity of light, Planck constant and the particle mass are unity. Time is divided in small equal steps of length ϵ. Reversals of path direction can occur only at the boundaries of these steps, that is at $t = t_a + n\epsilon$, where n is an integer. The amplitude to go along a path with R reversals is

$$\phi = (i\epsilon)^R.$$

Calculate the kernel $K(b, a)$ by adding together the contribution for the paths of one corner, two corners, and so on

$$K(b, a) = \sum_R N(R)(i\epsilon)^R$$

Let $A(x_a, t_a)$ and $B(x_b, t_b)$ be two events and a relativistic particle that moves from A to B. In order for the particle to reach B from A at an average velocity smaller than c we need $X = (x_b - x_a) < (t_b - t_a) = T$. If we divide the time interval $t_b - t_a$ in N_t steps of length ϵ, then

$$N_t = \frac{t_b - t_a}{\epsilon} \qquad \text{and} \qquad N_x = \frac{x_b - x_a}{\epsilon}$$

with N_x the number of steps in space between A and B.

We have then

$$N_- = \frac{1}{2}(T - X), \qquad N_+ = \frac{1}{2}(T + X)$$

are the positive steps, and negative steps necessary to reach B from A.

The total number of reversals is then given by

$$1 \leqslant R \leqslant 2r, \qquad r = \min(N_-, N_+)$$

An odd number of reversals change a positive motion into a negative one, and vice versa. An even number of reversals keeps the direction of motion. So we can write

$$K_{xy}(b, a) = \sum_{R=2m+1} (i\epsilon)^{2m+1} N(R)$$

where $(x, y) \in \{(+, -), (-, +)\}$, since only odd reversals are non-vanishing. Likewise

$$K_{xx}(b, a) = \sum_{R=2m} (i\epsilon)^{2m} N(R)$$

with $x \in \{+, -\}$. Clearly

$$N_{+-}(1) = N_{-+}(1) = 1 = \binom{N_+ - 1}{1 - 1} = \binom{N_- - 1}{0}, \qquad N_{--}(1) = N_{++}(1) = 0,$$

and

$$N_{+-}(2) = N_{-+}(2) = 0, \qquad N_{--}(2) = N_- - 1 = \binom{N_- - 1}{1}, \qquad N_{++}(2) = N_+ - 1 = \binom{N_+ - 1}{1}.$$

For $R = 3$, since in the K_{+-} kernel, the last step is the ℓ[1] turn that gives the $-$ direction, we have an ℓ turn that can take place among $N_+ - 1$ positions, and an r turn that can take place among $N_- - 1$ positions, so that

$$N_{+-}(3) = \binom{N_+ - 1}{1}\binom{N_- - 1}{1} = N_{-+}(3)$$

Say $n_\ell = \frac{R-1}{2} + 1 = \frac{R+1}{2}$ and $n_r = \frac{R-1}{2}$ the number of left and right turns. Then

$$N_{+-}(R) = \binom{N_+ - 1}{n_\ell - 1}\binom{N_- 1}{n_r}$$

since there are $N_+ - 1$ locations where the path can turn left, so we have to chose $n_\ell - 1$ of those (the last ℓ turn occurs at the end of the path) and we need to chose n_r right turns among $N_- - 1$ locations (we ignore the last position as it leads directly to the final point in the positive direction and not the negative one). So,

$$K_{+-}(b,a) = \sum_{m=1}^{M} \binom{N_+ - 1}{m}\binom{N_- - 1}{m} (i\epsilon)^{2m+1}$$

where $R = 2m + 1$, $n_\ell = m + 1$ and $n_r = m$ and $M = N_+$ is the maximal number of left turns.

As it happens we have

$$a! \simeq \sqrt{2\pi} a^{a+\frac{1}{2}} e^{-a}$$

so that

$$\binom{a}{m} \simeq \frac{a^{a+\frac{1}{2}}}{\sqrt{2\pi} m^{m+\frac{1}{2}} (a-m)^{a-m+\frac{1}{2}}}$$

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1. An ℓ turn is a left turn that changes a $+$ direction to a $-$ direction. An r turn does just the opposite.

Chapitre 3

Developing the Concepts with Special Examples

Chapitre 4

The Schrœdinger Description of Quantum Mechanics

Chapitre 5

Measurements and Operators

Annexe A

Rappels

A.1 Statistique 1D

Dans une longueur L les vecteurs d'état pour des états propres qui s'annulent aux extrémités sont

$$k_j = \frac{\pi}{L} j, \quad \forall j \in \mathbb{N}$$

et les impulsions

$$p_j = \hbar k_j = \frac{h}{2L} j = j \Delta p$$

avec $\Delta p = \frac{h}{2L}$.

De plus, $E^2 = p^2c^2 + m^2c^4$ et dans le régime non relativiste

$$E = mc^2 + \frac{p^2}{2m} + \ldots.$$

Ainsi $pc = \sqrt{E^2 - m^2c^4}$ et dans le régime non relativiste

$$p = \sqrt{2m(E - mc^2)}, \text{ avec } E \in [mc^2, +\infty[.$$

On a alors

$$dp = \sqrt{2m} \frac{dE}{2\sqrt{E - mc^2}}$$

La densité d'état entre p et $p + dp$ est

$$dn(p) = \frac{dN}{L} = \frac{1}{L} \frac{dp}{\Delta p} = \frac{2}{h} dp$$

et en fonction de l'énergie

$$dn(E) = \frac{\sqrt{2m}}{h} \frac{dE}{\sqrt{E - mc^2}}$$

1. **Statistique de Fermi-Dirac** Si chaque état est dégénéré g fois, et que le nombre total de particules est N et la densité de particule $n = \frac{N}{L}$, alors, pour des fermions d'énergie de Fermi E_F à $T = 0$ avec une distribution

$$f(E) = \begin{cases} 1 & \text{si } E < \mu - mc^2 \\ 0 & \text{si } E > \mu - mc^2 \end{cases}$$

et ave $\mu = E_F + mc^2$, pour $T = 0$, on trouve

$$n_F = g\frac{\sqrt{2m}}{h}\int_{mc^2}^{E_F}\frac{dE}{\sqrt{E - mc^2}}$$

soit

$$\boxed{n_F = 2g\frac{\sqrt{2m}}{h}\sqrt{E_F - mc^2}.}$$

La densité d'énergie pour des fermions à $T = 0$

$$\epsilon_F = \int_{mc^2}^{E_F} E\,dn(E) = \frac{2g}{h}\int_0^{p_F}(mc^2 + \frac{p^2}{2m})\,dp$$

soit

$$\boxed{\epsilon_F = \frac{2g}{3h}(E_F + 2mc^2)\sqrt{2m(E_F - mc^2)}.}$$

2. **Statistique de Boltzman** Pour une statistique de Boltzman, la distribution de Boltzman s'écrit

$$f(E) = e^{\beta(\mu - E)} = e^{\beta(\mu - mc^2 - \frac{p^2}{2m})}$$

on obtient, avec $\beta = \frac{1}{kT}$ et avec $K = \frac{p^2}{2m}$

$$n_B = g\frac{\sqrt{2m}}{h}\int_{mc^2}^{\infty}\frac{e^{\beta(\mu - mc^2 - K)}}{\sqrt{E - mc^2}}dE = g\frac{\sqrt{2m}}{h}e^{\beta(\mu - mc^2)}\int_0^{\infty}\frac{e^{-\beta K}}{\sqrt{K}}dK$$

en posant $t = \sqrt{K}$ on a

$$\int_0^{\infty}\frac{e^{-\beta K}}{\sqrt{K}}dK = \int_0^{\infty}\frac{e^{-\beta t^2}}{t}2t\,dt = 2\sqrt{\frac{\pi}{\beta}} = 2\sqrt{\pi kT}$$

Ainsi

$$\boxed{n_B = \frac{2g}{h}\sqrt{2\pi mkT}e^{\beta(\mu - mc^2)}.}$$

Enfin pour la densité d'énergie

$$\begin{aligned}\epsilon_B &= \int_{mc^2}^{\infty} E\,dn(E)\\ &= g\frac{\sqrt{2m}}{h}e^{\beta(\mu - mc^2)}\int_0^{\infty}(mc^2 + K)\frac{e^{-\beta K}}{\sqrt{K}}dK\\ &= g\frac{\sqrt{2m}}{2h}e^{\beta(\mu - mc^2)}\sqrt{\pi kT}(4mc^2 + kT)\end{aligned}$$

soit

$$\boxed{\epsilon_B = g\frac{\sqrt{2\pi mkT}}{2h}(4mc^2 + kT)e^{\beta(\mu - mc^2)}}$$

et

$$\frac{\epsilon_B}{n_B} = \frac{1}{4}(4mc^2 + kT) = mc^2 + \frac{1}{4}kT$$

A.2 Statistique 2D

Avec $p_2 = \hbar k_2 = \hbar \frac{\pi}{L}(n_x, n_y)$, pour $\{n_x, n_y\} \subset \mathbb{N}$ on a

$$p^2 = \hbar^2 k^2 = \frac{h^2}{4L}(n_x^2 + n_y^2) = \frac{h^2}{4L} n^2.$$

Le nombre d'état correspondant à une valeur entre n et $n + dn$ est

$$dN = \frac{1}{4} 2\pi n \, dn = \frac{1}{4} 2\pi \frac{2L}{h} p \frac{2L}{h} \, dp = \pi \frac{2A}{h^2} p \, dp$$

soit une densité d'état

$$dn(p) = \frac{2\pi}{h^2} p \, dp$$

En prenant en compte la dégénérescence g des états et le taux d'occupation $f(p)$

$$dn(p) = g \frac{2\pi}{h^2} p f(p) \, dp$$

et

$$dn(E) = g \frac{4\pi m}{h^2} f(E) \, dE$$

A.3 Statistique 3D

Avec $p = \hbar k$ et $k = \frac{\pi}{L}(n_x, n_y, n_z)$ on a

$$p^2 = \frac{h^2}{4L^2} n^2$$

et le nombre d'état entre n et $n + dn$

$$dN = \frac{1}{8} 4\pi n^2 dn = \frac{1}{8} 4\pi \frac{8L^3}{h^3} p^2 dp$$

d'où une densité d'état

$$dn(p) = \frac{4\pi}{h^3} p^2 dp$$

En prenant en compte la dégénérescence des niveaux et le taux d'occupation, on a

$$dn(p) = \frac{4\pi}{h^3} g f(p) p^2 dp$$

et

$$dn(E) = g \frac{2\pi}{h^3} (2m)^{\frac{3}{2}} \sqrt{E - mc^2} f(E) dE$$

1. **Statistique de Fermi-Dirac.** La densité de particules $T = 0$ devient

$$n_F = g \frac{4\pi}{h^3} m \int_0^{p_F} p^2 dp$$

soit

$$\boxed{n_F = g \frac{4\pi}{3h^3} p_F^3 = g \frac{4\pi}{3h^3} \left(2m(E_F - mc^2)\right)^{\frac{3}{2}}}$$

et la densité d'énergie

$$\epsilon_F = g\frac{2\pi}{h^3}(2m)^{\frac{3}{2}}\int_{mc^2}^{E_F} E\sqrt{E-mc^2}dE$$

d'où

$$\epsilon_F = g\frac{2\pi}{h^3}(2m)^{\frac{3}{2}}\frac{2}{15}(E_F-mc^2)^{\frac{3}{2}}(3E_F+2mc^2)$$

soit

$$\epsilon_F = g\frac{4\pi}{15h^3}(2m)^{\frac{3}{2}}(E_F-mc^2)^{\frac{3}{2}}(3E_F+2mc^2)$$

et

$$\boxed{\frac{\epsilon_F}{n_F} = \frac{2}{5}mc^2 + \frac{3}{5}E_F.}$$

2. **Statistique de Boltzman.** On obtient la densité de particules

$$n_B = g\frac{2\pi}{h^3}(2m)^{\frac{3}{2}}\int_0^\infty \sqrt{K}e^{\beta(\mu-mc^2-K)}dK$$

soit

$$n_B = g\frac{2\pi}{h^3}(2m)^{\frac{3}{2}}e^{\beta(\mu-mc^2)}\frac{1}{2}kT\sqrt{\pi kT}$$

$$\boxed{n_B = g\left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}}e^{\beta(\mu-mc^2)}}$$

et la densité d'énergie

$$\epsilon_B = g\frac{2\pi}{h^3}(2m)^{\frac{3}{2}}\int_0^\infty (mc^2+K)\sqrt{K}e^{\beta(\mu-mc^2-K)}dK$$

et par intégrations par parties on obtient

$$\epsilon_B = g\frac{2\pi}{h^3}(2m)^{\frac{3}{2}}\frac{1}{2}kT\sqrt{\pi kT}(mc^2+\frac{3}{2}kT)e^{\beta(\mu-mc^2)}$$

soit

$$\boxed{\epsilon_B = g\left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}}(mc^2+\frac{3}{2}kT)e^{\beta(\mu-mc^2)}}$$

d'où

$$\boxed{\frac{\epsilon_B}{n_B} = mc^2 + \frac{3}{2}kT.}$$

Calculons encore la pression

$$P = \frac{g}{3}\frac{4\pi}{mh^3}e^{\beta(\mu-mc^2)}\int_0^\infty p^4 e^{-\beta\frac{p^2}{2m}}dp$$

d'où

$$\begin{aligned} P &= g\frac{4\pi}{3mh^3}e^{\beta(\mu-mc^2)}\frac{3}{8}4m^2k^2T^2\sqrt{2\pi mkT} \\ &= g\left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}}kTe^{\beta(\mu-mc^2)} \end{aligned}$$

soit

$$\boxed{P = nkT.}$$

3. **Cas relativiste** Dans ce cas, on a

$$E^2 = p^2c^2 + m^2c^4$$

de sorte que

$$pc = \sqrt{E^2 - m^2c^4} = mc^2\sqrt{x^2 - 1}$$

et

$$dp = \frac{E\,dE}{\sqrt{E^2 - m^2c^4}} = mc^2\frac{x\,dx}{\sqrt{x^2 - 1}}$$

avec $x = \frac{E}{mc^2}$. On trouve alors

$$dn(x) = g\frac{4\pi}{c^2h^3}(mc^2)^3\sqrt{x^2 - 1}x\,dx$$