Quantum Field Theories :
Notes and exercises

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15 novembre 2017

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Première partie

Björken and Drell I. Relativistic Quantum Mechanics

Conventions

D'après les livres [BD64, BD65]. Les conventions sont les suivantes
Pour la métrique de Minkowski $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, (avec $\hbar = c = 1$), $p^0 = \frac{E}{c}$

$$p^\mu = (p^0, \mathbf{p}), \qquad \partial^\mu = (\partial_0, \nabla)$$

et

$$p_\mu = (p^0, -\mathbf{p}), \qquad \partial_\mu = (\partial_0, -\nabla).$$

Ainsi

$$p_\mu p^\mu = (p^0)^2 - \mathbf{p}^2 = m^2.$$

Pour les matrices de Dirac

$$\alpha = \begin{pmatrix} \mathbf{0} & \sigma \\ \sigma & \mathbf{0} \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$$

avec les matrices de Pauli $\sigma = (\sigma^1, \sigma^2, \sigma^3)$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad [\sigma^i, \sigma^j] = 2i\epsilon_{ijk}\sigma^k,$$

on obtient

$$\gamma^0 = \beta \quad \text{et} \quad \gamma^i = \beta\alpha^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}$$

soit

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

qui vérifient

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \qquad \{\gamma^\mu, \gamma^5\} = 0.$$

Finalement

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu], \qquad \sigma^{\mu\mu} = 0$$

et

$$\sigma^{0i} = i \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \qquad \sigma^{ij} = \epsilon_{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

Chapitre 1

The Dirac equation

1.1 Problems

1.1.1 Problem 1 p.13

Récrire les équations de Maxwell sous la forme d'une équation de Dirac de la forme

$$i\hbar\partial_t\psi = \frac{\hbar c}{i}\alpha^j\partial_j\psi + \beta mc^2\psi,$$

où ψ est une amplitude à six composantes. Que valent les matrices correspondant aux matrices α et β ?

En unités CGS, les équations de Maxwell s'écrivent

$$\begin{cases} \nabla \cdot E & = 4\pi\rho \\ \nabla \cdot B & = 0 \\ \partial_t B + \nabla \wedge E & = 0 \\ \partial_t E - \nabla \wedge B & = 4\pi j \end{cases}$$

Les deux première équations découlent de l'équation de continuité et des deux dernières équations.

Ces deux dernières équations peuvent se mettre sous la forme de l'équation de Dirac sans termes de masse ($\beta = 0$)

$$\partial_t F + \sum_{i=1}^{3} \alpha^i \partial_i F = J$$

où $F = (E, B)$ et $J = (j, 0)$ sont des vecteurs à 6 composantes et les α^i sont des matrices 6×6. ce qui donne

$$\partial_t F + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \partial_1 F + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \partial_2 F$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \partial_3 F = J$$

1.1.2 Problem 2 p.13

Montrer que les matrices

$$\alpha^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ \sigma^i & \mathbf{0} \end{pmatrix} \qquad \textbf{et} \qquad \beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}$$

vérifient l'algèbre

$$\{\alpha^i, \alpha^k\} = 2\delta^{ik}, \qquad \{\beta, \alpha^i\} = 0, \quad \text{et} \quad \beta^2 = \alpha_i^2 = 1.$$

Par calcul direct

$$\alpha^i\alpha^j = \begin{pmatrix} \sigma^i\sigma^j & \mathbf{0} \\ \mathbf{0} & \sigma^i\sigma^j \end{pmatrix}$$

de sorte que

$$\{\alpha^i, \alpha^j\} = \begin{pmatrix} \{\sigma^i, \sigma^j\} & \mathbf{0} \\ \mathbf{0} & \{\sigma^i, \sigma^j\} \end{pmatrix} = 2\delta_{ij}\mathbf{1}_{4\times 4},$$

car $\{\sigma^i, \sigma^j\} = 2\delta_{ij}$.

De plus

$$\beta\alpha^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}$$

et

$$\alpha^i\beta = \begin{pmatrix} \mathbf{0} & -\sigma^i \\ \sigma^i & \mathbf{0} \end{pmatrix}$$

ainsi

$$\{\beta, \alpha^i\} = 0.$$

Enfin, on voit clairement que

$$\beta^2 = 1.$$

1.1.3 Problem 3 p.13

Montrer qu'avec $\pi = p - \frac{e}{c}A$ on a

$$(\sigma \cdot \pi)^2 = \pi^2 - \frac{e\hbar}{c}\sigma \cdot \mathbf{B}.$$

Clairement on a

$$\begin{aligned}(\sigma \cdot a)(\sigma \cdot b) &= \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}\begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix} \\ &= \begin{pmatrix} a \cdot b + i(a_1b_2 - a_2b_1) & (a_3b_1 - a_1b_3) - i(a_3b_2 - a_2b_3) \\ -(a_3b_1 - a_1b_3) + i(a_2b_3 - a_3b_2) & a \cdot b + i(a_2b_1 - a_1b_2) \end{pmatrix}\end{aligned}$$

soit

$$\boxed{(\sigma \cdot a)(\sigma \cdot b) = a \cdot b + i(a \wedge b) \cdot \sigma.}$$

Ainsi

$$(\sigma \cdot \pi)(\sigma \cdot \pi) = \pi \cdot \pi + i(\pi \wedge \pi) \cdot \sigma.$$

Or, on a

$$\begin{aligned}\pi \wedge \pi &= (p - \frac{e}{c}A) \wedge (p - \frac{e}{c}A) \\ &= -\frac{e}{c}(p \wedge A + A \wedge p) \\ &= -\frac{e\hbar}{ic}\left[\begin{pmatrix} \partial_2 A_3 - \partial_3 A_2 \\ \vdots \end{pmatrix} + A \wedge \nabla\right] \\ &= -\frac{e\hbar}{ic}\left[\begin{pmatrix} (\partial_2 A_3) - (\partial_3 A_2) + A_3\partial_2 - A_2\partial_3 \\ \vdots \end{pmatrix} + A \wedge \nabla\right] \\ &= -\frac{e\hbar}{ic}\left[(\nabla \wedge A) - A \wedge \nabla + A \wedge \nabla\right] \\ &= -\frac{e\hbar}{ic}B\end{aligned}$$

Ainsi, on obtient

$$\boxed{(\sigma \cdot \pi)(\sigma \cdot \pi) = \pi \cdot \pi - \frac{e\hbar}{c}B \cdot \sigma.}$$

1.1.4 Problem 4 p.13

Vérifier que l'on a les relations d'Ehrenfest

$$d_t\mathbf{r} = \frac{i}{\hbar}[H,\mathbf{r}] = c\alpha \equiv \mathbf{v}_{op}$$
$$d_t\pi = \frac{i}{\hbar}[H,\pi] - \frac{e}{c}\partial_t A = e\big[E + \frac{1}{c}\mathbf{v}_{op}\times\mathbf{B}\big].$$

L'hamiltonien de Dirac s'écrit

$$H = c\alpha\cdot(p-\frac{e}{c}A) + \beta mc^2 + e\phi.$$

On a alors,

$$\begin{aligned}
\frac{d}{dt}r &= \frac{i}{\hbar}[H,r]\\
&= \frac{i}{\hbar}[c\alpha\cdot(p-\frac{e}{c}A) + \beta mc^2 + e\phi, r]\\
&= \frac{i}{\hbar}[c\alpha\cdot p, r]\\
&= \frac{i}{\hbar}c\alpha\cdot[p,r]
\end{aligned}$$

avec $[p,r] = -i\hbar$ on trouve

$$\boxed{\frac{d}{dt}r = c\alpha \equiv \mathbf{v}_{op}.}$$

De plus

$$\begin{aligned}
\frac{d}{dt}\pi &= \frac{i}{\hbar}[H,\pi] - \frac{e}{c}\partial_t A\\
&= \frac{i}{\hbar}[c\alpha\cdot(p-\frac{e}{c}A) + \beta mc^2 + e\phi, p - \frac{e}{c}A] - \frac{e}{c}\partial_t A\\
&= \frac{i}{\hbar}\big([c\alpha\cdot(p-\frac{e}{c}A), p-\frac{e}{c}A] + [e\phi, p-\frac{e}{c}A]\big) - \frac{e}{c}\partial_t A\\
&= \frac{i}{\hbar}\big([c\alpha\cdot(p-\frac{e}{c}A), p] - \frac{e}{c}[c\alpha\cdot(p-\frac{e}{c}A), A] + [e\phi, p]\big) - \frac{e}{c}\partial_t A\\
&= \frac{i}{\hbar}\big([c\alpha\cdot(p-\frac{e}{c}A), p] - \frac{e}{c}[c\alpha\cdot(p-\frac{e}{c}A), A]\big) + e[\phi\nabla - \nabla\phi] - \frac{e}{c}\partial_t A\\
&= -\frac{e}{c}[c\alpha\cdot A, \nabla] - \frac{e}{c}\frac{i}{\hbar}[c\alpha\cdot(p-\frac{e}{c}A), A] + e(\phi\nabla - (\nabla\phi) - \phi\nabla) - \frac{e}{c}\partial_t A\\
&= -e\alpha^i\cdot[A_i,\nabla] - e\alpha^i\cdot[\partial_i, A] + e\big(-\nabla\phi - \frac{1}{c}\partial_t A\big)\\
&= -e\alpha^i\cdot\big([A_i,\nabla] + [\partial_i, A]\big) + eE\\
&= -\frac{e}{c}\mathbf{v}^i_{op}\cdot\big([A_i,\nabla] + [\partial_i, A]\big) + eE
\end{aligned}$$

Avec

$$\begin{aligned}
F_i = [A_i,\nabla] + [\partial_i, A] &= A_i\nabla - \nabla A_i + \partial_i A - A\partial_i\\
&= -(\nabla A_i) + (\partial_i A)\\
&= -\begin{pmatrix}\partial_1 A_i\\ \partial_2 A_i\\ \partial_3 A_i\end{pmatrix} + \begin{pmatrix}\partial_i A_1\\ \partial_i A_2\\ \partial_i A_3\end{pmatrix}\\
&= \begin{pmatrix}\partial_i A_1 - \partial_1 A_i\\ \partial_i A_2 - \partial_2 A_i\\ \partial_i A_3 - \partial_3 A_i\end{pmatrix}
\end{aligned}$$

on obtient

$$F_1 = \begin{pmatrix}0\\ \partial_1 A_2 - \partial_2 A_1\\ \partial_1 A_3 - \partial_3 A_1\end{pmatrix} = \begin{pmatrix}0\\ B_3\\ -B_2\end{pmatrix}$$

$$F_2 = \begin{pmatrix} \partial_2 A_1 - \partial_1 A_2 \\ 0 \\ \partial_2 A_3 - \partial_3 A_2 \end{pmatrix} = \begin{pmatrix} -B_3 \\ 0 \\ B_1 \end{pmatrix}$$

et

$$F_3 = \begin{pmatrix} \partial_3 A_1 - \partial_1 A_3 \\ \partial_3 A_2 - \partial_2 A_3 \\ 0 \end{pmatrix} = \begin{pmatrix} B_2 \\ -B_1 \\ 0 \end{pmatrix}.$$

Ainsi, on a

$$v \cdot F = v^i F_i = \begin{pmatrix} v^3 B_2 - v^2 B_3 \\ v^1 B_3 - v^3 B_1 \\ v^2 B_1 - v^1 B_2 \end{pmatrix} = -v \wedge B,$$

et

$$\frac{d}{dt}\pi = \frac{e}{c}\mathbf{v}_{op} \wedge B + eE.$$

soit

$$\boxed{\frac{d}{dt}\pi = e\big(E + \frac{1}{c}\mathbf{v}_{op} \wedge B\big).}$$

1.2 Pauli equation p.13

Dirac equation

$$i\hbar\partial_t \psi = \left[c\alpha \cdot (p - \frac{e}{c}A) + \beta mc^2 + e\phi\right]\psi$$

with

$$\psi = \begin{pmatrix} \tilde{\chi} \\ \tilde{\xi} \end{pmatrix}$$

$$i\hbar\partial_t \begin{pmatrix} \tilde{\chi} \\ \tilde{\xi} \end{pmatrix} = c(p - \frac{e}{c}A) \cdot \begin{pmatrix} \sigma\tilde{\xi} \\ \sigma\tilde{\chi} \end{pmatrix} + mc^2 \begin{pmatrix} \tilde{\chi} \\ -\tilde{\xi} \end{pmatrix} + e\phi \begin{pmatrix} \tilde{\chi} \\ \tilde{\xi} \end{pmatrix}$$

With positive energy and χ and ξ slowly varying functions of time

$$\begin{pmatrix} \tilde{\chi} \\ \tilde{\xi} \end{pmatrix} = e^{-i\frac{mc^2}{\hbar}t} \begin{pmatrix} \chi \\ \xi \end{pmatrix}$$

we obtains, with $\pi = p - \frac{e}{c}A$

$$i\hbar\partial_t \begin{pmatrix} \chi \\ \xi \end{pmatrix} = c\sigma \cdot \pi \begin{pmatrix} \xi \\ \chi \end{pmatrix} + e\phi \begin{pmatrix} \chi \\ \xi \end{pmatrix} - 2mc^2 \begin{pmatrix} 0 \\ \xi \end{pmatrix}$$

that is

$$i\hbar\partial_t \chi = c\sigma \cdot \pi\xi + e\phi\chi \qquad (*)$$
$$i\hbar\partial_t \xi = c\sigma \cdot \pi\chi + e\phi\xi - 2mc^2\xi$$

For slowly varying ξ

$$0 = c\sigma \cdot \pi\chi + e\phi\xi - 2mc^2\xi$$

and neglecting $e\phi\xi$ compared to $2mc^2\xi$

$$\xi = \frac{\sigma \cdot \pi}{2mc}\chi$$

and in $(*)$

$$i\hbar\partial_t \chi = \left[\frac{(\sigma \cdot \pi)(\sigma \cdot \pi)}{2m} + e\phi\right]\chi$$

and χ is a 2 component spinor.

$$(\sigma \cdot \pi)(\sigma \cdot \pi) = \pi^2 + i\sigma\pi \times \pi$$
$$= \pi^2 - \frac{e\hbar}{c}\sigma \cdot B.$$

We get then the Pauli equation

$$\boxed{i\hbar\partial_t\chi = \left[\frac{(p-\frac{e}{c}A)^2}{2m} - \frac{e\hbar}{2mc}\sigma\cdot B + e\phi\right]\chi}$$

Also

$$(p-\frac{e}{c}A)^2 = p^2 - \frac{e}{c}(p\cdot A + A\cdot p) + o(A^2)$$

to first order in the fields.

With

$$A = \frac{1}{2}B\times r = \frac{1}{2}\begin{pmatrix} B_2r_3 - B_3r_2 \\ B_3r_1 - B_1r_3 \\ B_1r_2 - B_2r_1 \end{pmatrix}$$

we have

$$p\cdot A = -\frac{1}{2}i\hbar r\cdot(\nabla\times B) + \frac{1}{2}B\cdot(r\times p) = -\frac{1}{2}i\hbar r\cdot(\nabla\times B) + \frac{1}{2}B\cdot L$$

and

$$A\cdot p = \frac{1}{2}B\cdot L$$

so that

$$p\cdot A + A\cdot p = -\frac{1}{2}i\hbar r\cdot(\nabla\times B) + B\cdot L.$$

Then, for a **uniform weak field** $\nabla\times B = 0$

$$i\hbar\partial_t\chi = \left[\frac{p^2}{2m} - \frac{e}{2mc}\left(L+\hbar\sigma\right)\cdot B + e\phi\right]\chi$$

and with the spin operator $S = \frac{1}{2}\hbar\sigma$ we recover the Pauli equation with the gyromagnetic moment for the spin $g = 2$

$$\boxed{i\hbar\partial_t\chi = \left[\frac{p^2}{2m} - \frac{e}{2mc}\left(L+2S\right)\cdot B + e\phi\right]\chi}$$

Chapitre 2

Lorentz Covariance of the Dirac equation

Exercise 2.1

Verify (2.26)

Eq (2.26) states that

$$S^{-1} = \gamma_0 S^\dagger \gamma_0$$

with (2.22)

$$S = \exp\left(-\frac{i}{4}\omega\sigma_{\mu\nu}I_n^{\mu\nu}\right)$$

and, by (2.16)

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

and using $[\gamma_0, \sigma^{\mu\nu}] = 0$

Exercise 2.2

Verify the transformation laws given in (2.38)

The laws given in (2.38) are

$$\begin{cases} \bar{\psi}'(x')\psi'(x') & = \bar{\psi}(x)\psi(x) \\ \bar{\psi}'(x')\gamma_5\psi'(x') & = \bar{\psi}(x)S^{-1}\gamma_5 S\psi(x) = \det(a)\bar{\psi}(x)\gamma_5\psi(x) \\ \bar{\psi}'(x')\gamma^\nu\psi'(x') & = a^\nu{}_\mu\bar{\psi}(x)\gamma^\mu\psi(x) \\ \bar{\psi}'(x')\gamma_5\gamma^\nu\psi'(x') & = \det(a)a^\nu{}_\mu\bar{\psi}(x)\gamma_5\gamma^\mu\psi(x) \\ \bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') & = a^\mu{}_\alpha a^\nu{}_\beta\bar{\psi}(x)\sigma^{\alpha\beta}\psi(x) \end{cases}$$

Chapitre 3

Solutions to the Dirac Equation for a Free Particle

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The Foldy-Wouthuysen Transformation

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Nonelectromagnetic Interactions

Deuxième partie

Björken and Drell II. Relativistic Quantum Fields

Chapitre 11

Quantum Fields : General Formalism

11.1 Problems : Le champs vectoriel massif

On part de l'équation d'onde du boson massif de spin 1

$$\left[g_{\mu\nu}(\Box + m^2) - \partial_\mu\partial_\nu\right]\phi^\nu(x) = 0,$$

d'où l'on déduit la contrainte

$$\partial_\nu\phi^\nu = 0.$$

a) Construction de la densité lagrangienne.
Supposons qu'il s'agit d'une équation d'Euler-Lagrange, on obtient la forme variationnelle en multipliant l'équation ci-dessus par $\delta\phi^\mu$

$$g_{\mu\nu}\partial_\alpha\partial^\alpha\phi^\nu\delta\phi^\mu + m^2 g_{\mu\nu}\phi^\nu\delta\phi^\mu - \partial_\mu\partial_\nu\phi^\nu\delta\phi^\mu = 0,$$

d'où

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi^\mu}\delta\phi^\mu = g_{\mu\nu}\partial_\alpha\partial^\alpha\phi^\nu\delta\phi^\mu - \partial_\mu\partial_\nu\phi^\nu\delta\phi^\mu + m^2 g_{\mu\nu}\phi^\nu\delta\phi^\mu.$$

Le terme de masse provient clairement de la variation de

$$\mathcal{L}_m = \frac{1}{2}m^2 g_{\mu\nu}\phi^\mu\phi^\nu = \frac{1}{2}m^2\phi^\nu\phi_\nu.$$

La partie cinétique s'obtient en intégrant une fois par partie les termes avec dérivées

$$\delta\mathcal{L}' = -g_{\mu\nu}\partial^\alpha\phi^\nu\partial_\alpha\delta\phi^\mu + \partial_\nu\phi^\nu\partial_\mu\delta\phi^\mu.$$

En utilisant la propriété $\partial_\alpha\delta\phi = \delta(\partial_\alpha\phi)$ on obtient

$$\delta\mathcal{L}' = -g_{\mu\nu}\partial^\alpha\phi^\nu\delta(\partial_\alpha\phi^\mu) + \partial_\nu\phi^\nu\delta(\partial_\mu\phi^\mu),$$

qui provient clairement de

$$\mathcal{L}' = -\frac{1}{2}g_{\mu\nu}(\partial^\alpha\phi^\nu)(\partial_\alpha\phi^\mu) + \frac{1}{2}(\partial_\nu\phi^\nu)(\partial_\mu\phi^\mu)$$

que l'on peut récrire

$$\mathcal{L}' = -\frac{1}{2}(\partial^\alpha\phi^\nu)(\partial_\alpha\phi_\nu) + \frac{1}{2}(\partial_\nu\phi^\nu)^2.$$

En remplaçant α par μ on peut écrire le lagrangien

$$\boxed{\mathcal{L} = -\frac{1}{2}(\partial^\mu\phi^\nu)(\partial_\mu\phi_\nu) + \frac{1}{2}m^2\phi^\nu\phi_\nu + \frac{1}{2}(\partial_\nu\phi^\nu)^2}$$

b) Construction de la densité hamiltonienne.
Les moment conjugués sont donnés par

$$\pi^\nu = \frac{\delta\mathcal{L}}{\delta(\partial_0\phi^\nu)} = -\partial_0\phi^\nu + \delta_{0\nu}\partial_\mu\phi^\mu.$$

Ainsi, en prenant en compte que $\partial_0\phi^\nu\partial_0\phi_\nu = \partial_0\phi^0\partial_0\phi^0 - \partial_0\phi^j\partial_0\phi_j$, on obtient

$$\begin{cases} \pi^0 & = -\partial_0\phi^0 + \partial_\mu\phi^\mu = \partial_j\phi^j \\ \pi^j & = \partial_0\phi^j. \end{cases}$$

En termes de π^μ la densité lagrangienne s'écrit

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}(\partial^0\phi^\nu)(\partial_0\phi_\nu) - \frac{1}{2}(\partial^j\phi^\nu)(\partial_j\phi_\nu) + \frac{1}{2}(\partial_0\phi^0)(\partial_0\phi^0) \\ & + (\partial_0\phi^0)(\partial_j\phi^j) + \frac{1}{2}(\partial_i\phi^i)(\partial_j\phi^j) + \frac{1}{2}m^2\phi_\nu\phi^\nu \\ = & -\frac{1}{2}\partial_0\phi^0\partial_0\phi^0 + \frac{1}{2}\pi_j\pi^j - \frac{1}{2}(\nabla\phi_\nu)\cdot(\nabla\phi^\nu) + \frac{1}{2}(\partial_0\phi^0)^2 + \partial_0\phi^0\pi^0 \\ & + \frac{1}{2}(\pi^0)^2 + \frac{1}{2}m^2\phi_\nu\phi^\nu \\ = & \frac{1}{2}\pi_j\pi^j - \frac{1}{2}(\nabla\phi_\nu)\cdot(\nabla\phi^\nu) + \partial_0\phi^0\pi^0 \\ & + \frac{1}{2}(\pi^0)^2 + \frac{1}{2}m^2\phi_\nu\phi^\nu \end{aligned}$$

avec la contrainte

$$\partial_\nu\phi^\nu = 0 \qquad \Longleftrightarrow \qquad \partial_0\phi^0 = -\partial_j\phi^j = -\pi^0.$$

Ainsi

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}\pi_j\pi^j - \frac{1}{2}(\nabla\phi_\nu)\cdot(\nabla\phi^\nu) + \frac{1}{2}(\pi^0)^2 - \pi^0\pi^0 \\ & + \frac{1}{2}(\pi^0)^2 + \frac{1}{2}m^2\phi_\nu\phi^\nu \\ = & -\frac{1}{2}(\pi^0)^2 + \frac{1}{2}\pi_j\pi^j - \frac{1}{2}(\nabla\phi_\nu)\cdot(\nabla\phi^\nu) + \frac{1}{2}m^2\phi_\nu\phi^\nu \\ = & -\frac{1}{2}\pi_\nu\pi^\nu - \frac{1}{2}(\nabla\phi_\nu)\cdot(\nabla\phi^\nu) + \frac{1}{2}m^2\phi_\nu\phi^\nu \end{aligned}$$

La densité hamiltonienne s'écrit

$$\begin{aligned} \mathcal{H} & = \pi_\mu\partial_0\phi^\mu - \mathcal{L} \\ & = \pi_0(-\pi^0) + \pi_j\pi^j + \frac{1}{2}\pi_\nu\pi^\nu + \frac{1}{2}(\nabla\phi_\nu)\cdot(\nabla\phi^\nu) - \frac{1}{2}m^2\phi_\nu\phi^\nu \\ & = -\frac{1}{2}(\pi^0)^2 + \frac{1}{2}\pi_j\pi^j - \frac{1}{2}(\nabla\phi_\nu)\cdot(\nabla\phi^\nu) - \frac{1}{2}m^2\phi_\nu\phi^\nu \end{aligned}$$

On obtient finalement

$$\boxed{\mathcal{H} = -\frac{1}{2}\pi_\nu\pi^\nu - \frac{1}{2}(\nabla\phi_\nu)\cdot(\nabla\phi^\nu) - \frac{1}{2}m^2\phi_\nu\phi^\nu.}$$

c)
d)

Troisième partie

Quigg

Conventions

D'après le livre de Quigg [Qui97]. Les conventions sont les suivantes
Pour la métrique de Minkowski (avec $\hbar = c = 1$)

$$p^\mu = (E, \mathbf{p}), \qquad \partial^\mu = (\partial_0, \nabla)$$

et

$$p_\mu = (E, -\mathbf{p}), \qquad \partial_\mu = (\partial_0, -\nabla).$$

Ainsi

$$p_\mu p^\mu = E^2 - \mathbf{p}^2 = m^2.$$

Chapitre 1

Introduction

Chapitre 2

Lagrangian Formalism and Conservation Laws

Quatrième partie

Aitchison and Hey

Conventions

D'après le livre de Aitchison et Hey [AH93]. Les conventions sont les suivantes

Pour la métrique de Minkowski (avec $\hbar = c = 1$)

$$p^\mu = (E, \mathbf{p}), \qquad \partial^\mu = (\partial_0, \nabla)$$

et

$$p_\mu = (E, -\mathbf{p}), \qquad \partial_\mu = (\partial_0, -\nabla).$$

Ainsi

$$p_\mu p^\mu = E^2 - \mathbf{p}^2 = m^2.$$

Pour les matrices de Dirac

$$\alpha = \begin{pmatrix} \mathbf{0} & \sigma \\ \sigma & \mathbf{0} \end{pmatrix}$$

et

$$\beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$$

Largeur de désintégration

$$d\Gamma = \frac{1}{2M} |\mathcal{M}|^2 \prod_{\text{out}} \left(\frac{d^3 p_i}{(2\pi)^3 2E_i} \right) (2\pi)^4 \delta(M - \sum_{\text{out}} p)$$

Section efficace

$$d\sigma = \frac{|\mathcal{M}|^2}{4[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{\frac{1}{2}}} \prod_{\text{out}} \left(\frac{d^3 p_i}{(2\pi)^3 2E_i} \right) (2\pi)^4 \delta \left(\sum_{\text{in}} p_i - \sum_{\text{out}} p_i \right)$$

Collisions élastiques $(e(k_1) + l(p_1) \to e(k_2) + l(p_2))$, variables de Mandelstam

$$s = (k_1 + p_1)^2 = (k_2 + p_2)^2$$

$$t = q^2 = (k_2 - k_1)^2 = (p_1 - p_2)^2$$

$$u = (k_2 - p_1)^2 = (p_2 - k_1)^2$$

$$s + t + u = \sum_i m_i^2.$$

Chapitre 2

Electromagnetism as a gauge theory

2.0.1 Problem 2.1 p.61

a) un boost dans la direction x^1 s'écrit, avec $c = 1$

$$\begin{cases} t' & = \gamma(t - vx^1) \\ x'^1 & = \gamma(-vt + x^1) \\ x'^2 & = x^2 \\ x'^3 & = x^3. \end{cases}$$

où $\gamma = (1 - v^2)^{-\frac{1}{2}}$.

b)

2.0.2 Problem 2.2 p.61

Déterminer les composantes indépendantes de $F^{\mu\nu}$.

2.0.3 Problem 2.3 p.61

Montrer que

$$e^{iqf(x)}\hat{p}e^{-iqf(x)} = \hat{p} - q\partial_x f(x)$$

On a

$$\begin{aligned} e^{iqf(x)}\hat{p}e^{-iqf(x)} &= e^{iqf(x)}(-i\partial_x)e^{-iqf(x)} \\ &= e^{iqf(x)}\big(- i(\partial_x e^{-iqf(x)}) - ie^{-iqf(x)}\partial_x\big) \\ &= e^{iqf(x)}e^{-iqf(x)}\big(- i(-iq\partial_x f(x) - i\partial_x\big) \\ &= -q\partial_x f(x) + \hat{p}. \end{aligned}$$

Ainsi

$$\boxed{e^{iqf(x)}\hat{p}e^{-iqf(x)} = \hat{p} - q\partial_x f(x).}$$

Chapitre 3

Klein-Gordon and Dirac equation

3.7 Problems

3.7.1 Problem 3.1 p.80

a) L'équation de Schrœdinger s'écrit

$$-\nabla^2\psi + V\psi - i\partial_t\psi = 0$$

et la forme conjuguée

$$-\nabla^2\psi^* + \psi^* V + i\partial_t\psi^* = 0.$$

En multipliant la première par ψ^* et la seconde par ψ on obtient respectivement

$$-\psi^*\nabla^2\psi + V\psi^*\psi - i\psi^*\partial_t\psi = 0$$

et

$$-\psi\nabla^2\psi^* + V\psi\psi^* + i\psi\partial_t\psi^* = 0.$$

La différence entre la deuxième et la première équation donne

$$\psi^*\nabla^2\psi - \psi\nabla^2\psi^* + i(\psi^*\partial_t\psi + \psi^*\partial_t\psi^*) = 0,$$

que l'on peut récrire

$$-i\nabla(\psi^*\nabla\psi - \psi\nabla\psi^*) + \partial_t(\psi^*\psi) = 0.$$

En posant

$$\rho = \psi^*\psi \qquad \text{et} \qquad j = \frac{1}{i}(\psi^*\nabla\psi - \psi\nabla\psi^*)$$

on obtient l'équation de continuité

$$\partial_t\rho + \nabla\cdot j = 0.$$

b) En partant de l'équation de Klein-Gordon

$$(\Box + m^2)\phi = 0$$

et de son conjugué

$$(\Box + m^2)\phi^* = 0$$

en procédant de même que précédemment on trouve

$$\phi^*\Box\phi + m^2\phi^*\phi = 0$$

et

$$\phi\Box\phi^* + m^2\phi^*\phi = 0.$$

En prenant la différence de la première avec la seconde équation on obtient

$$\phi^*\partial_\mu\partial^\mu\phi - \phi\partial_\mu\partial^\mu\phi^* = 0.$$

On peut sortir une dérivée partielle (la somme étant toujours sous-entendue)

$$\partial_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) = 0,$$

car on voit que les termes en $\partial_\mu \phi \partial^\mu \phi^*$ s'annulent.
Ainsi, le courant conservé est donné par, on divise par i pour obtenir la mÃªme forme que dans le cas du courant conservé de l'équation de Scrœdinger,

$$j^\mu = (\rho, \mathbf{j}) = \frac{1}{i}(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*).$$

Soit l'état libre

$$\phi = N e^{-ipx}$$

avec $p^\mu = (E, \mathbf{p})$ et $x^\mu = (t, \mathbf{r})$. Le courant conservé devient alors

$$j^\mu = \frac{1}{i} N^2 (e^{ipx}(-ip^\mu)e^{-ipx} - e^{-ipx}(ip^\mu)e^{ipx}) = 2N^2 p^\mu.$$

On voit que, dans ce cas, le courant conservé est proportionnel à p^μ.

3.7.2 Problem 3.2 p.81

a) Avec les matrices de Pauli 2×2

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{et} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

On a clairement

$$\sigma_i^2 = 1_{2\times 2} = \delta_{ii} 1_{2\times 2}$$

et

$$\sigma_1\sigma_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \text{et} \qquad \sigma_2\sigma_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

et

$$\sigma_2\sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \text{et} \qquad \sigma_3\sigma_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

et enfin

$$\sigma_3\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{et} \qquad \sigma_1\sigma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

On voit immédiatement que

$$\{\sigma_1, \sigma_2\} = \{\sigma_2, \sigma_3\} = \{\sigma_3, \sigma_1\} = 0$$

et avec l'identité sur σ_i^2 on a bien

$$\boxed{\{\sigma_i, \sigma_j\} = 2\delta_{ij} 1_{2\times 2}.}$$

De plus, on a

$$[\sigma_1, \sigma_2] = 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\sigma_3$$

$$[\sigma_2, \sigma_3] = 2i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2i\sigma_1$$

$$[\sigma_1, \sigma_2] = 2i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 2i\sigma_2$$

ce que l'on peut écrire de manière plus compacte

$$\boxed{[\sigma_i, \sigma_j] = 2i\epsilon^{ijk}\sigma_k.}$$

On voit alors, par la relation d'anticommutation, que

$$\sigma_j\sigma_i = 2\delta_{ij} - \sigma_i\sigma_j.$$

Dans la relation de commutation

$$\sigma_i\sigma_j - (2\delta_{ij} - \sigma_i\sigma_j) = 2i\epsilon^{ijk}\sigma_k.$$

Soit

$$\boxed{\sigma_i\sigma_j = \delta_{ij} + i\epsilon^{ijk}\sigma_k.}$$

b) Ainsi, on a (somme sur i, j et k sous-entendue)

$$\begin{aligned}(\sigma\cdot\mathbf{a})(\sigma\cdot\mathbf{b}) &= \sigma_i a^i \sigma_j b^j \\ &= a^i b^j \sigma_i \sigma_j \\ &= a^i b^j (\delta_{ij} + i\epsilon^{ijk}\sigma_k) \\ &= a^i b_i + i\epsilon^{ijk} a^i b^j \sigma_k \\ &= \mathbf{a}\cdot\mathbf{b} + i\sigma_{\mathbf{k}}\cdot(\mathbf{a}\times\mathbf{b}),\end{aligned}$$

soit

$$\boxed{(\sigma\cdot\mathbf{a})(\sigma\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} + i\sigma\cdot(\mathbf{a}\times\mathbf{b}).}$$

Avec

$$(\sigma\cdot\mathbf{p}) = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix},$$

on a

$$\begin{aligned}(\sigma\cdot\mathbf{p})^2 &= \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}^2 \\ &= \begin{pmatrix} p_z^2 + p_x^2 + p_y^2 & p_z(p_x - ip_y) - p_z(p_x - ip_y) \\ (p_x + ip_y)p_z - p_z(p_x + ip_y) & p_x^2 + p_y^2 + p_z^2 \end{pmatrix} \\ &= \begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix} = p^2 1_{2\times 2}.\end{aligned}$$

3.7.3 Problem 3.3 p.81

Soit la solution de l'équation de Dirac libre

$$\psi = \omega e^{-ipx}$$

avec le spineur ω défini par les spineurs Ã deux composantes ϕ et χ par

$$\omega = \begin{pmatrix} \phi \\ \chi \end{pmatrix}.$$

Dans l'équation de Dirac, on obtient

$$i\partial_t \psi = (-i\alpha\cdot\nabla + \beta m)\psi,$$

soit

$$i(-iE)\omega e^{-ipx} = \left[-i\begin{pmatrix} \mathbf{0} & \sigma \\ \sigma & \mathbf{0} \end{pmatrix}\cdot\nabla + \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}\right]\omega e^{-ipx}$$

soit, avec $px = Et - \mathbf{p}\cdot\mathbf{r}$

$$E\begin{pmatrix} \phi \\ \chi \end{pmatrix} e^{-ipx} = e^{-ipx}\left[-i\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}(-i(-p^j)) + \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}\right]\begin{pmatrix} \phi \\ \chi \end{pmatrix}.$$

Ainsi

$$E\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \left[\begin{pmatrix} m & \sigma_j p^j \\ \sigma_j p^j & -m \end{pmatrix}\right]\begin{pmatrix} \phi \\ \chi \end{pmatrix},$$

ce qui donne

$$\begin{pmatrix} E\phi \\ E\chi \end{pmatrix} = \begin{pmatrix} m\phi + (\sigma\cdot\mathbf{p})\chi \\ (\sigma\cdot\mathbf{p})\chi - m\phi \end{pmatrix},$$

soit

$$\begin{cases} (E-m)\phi &= (\sigma\cdot\mathbf{p})\chi \\ (E+m)\chi &= (\sigma\cdot\mathbf{p})\phi. \end{cases}$$

3.7.4 Problem 3.4 p.82

a) Les matrices α et β sont définies par

$$\alpha = \begin{pmatrix} \mathbf{0} & \sigma \\ \sigma & \mathbf{0} \end{pmatrix} \qquad \text{et} \qquad \beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$$

On a clairement $\beta^2 = 1_{2\times 2}$ et

$$\beta\alpha_j = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \sigma_j \\ -\sigma_j & \mathbf{0} \end{pmatrix}$$

et

$$\alpha_j\beta = \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & -\sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix}$$

de sorte que

$$\{\beta, \alpha_j\} = 0.$$

Enfin, on a

$$\alpha_i\alpha_j = \begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \sigma_i\sigma_j \\ \sigma_i\sigma_j & \mathbf{0} \end{pmatrix}$$

de sorte que

$$\{\alpha_i, \alpha_j\} = \begin{pmatrix} \mathbf{0} & \{\sigma_i, \sigma_j\} \\ \{\sigma_i, \sigma_j\} & \mathbf{0} \end{pmatrix} = 2\delta_{ij}1_{4\times 4}.$$

On définit, maintenant, les matrices de Dirac $\gamma^\mu = (\gamma^0, \gamma^j) = (\beta, \beta\alpha)$
On a alors, clairement

$$(\gamma^0)^2 = 1_{4\times 4}, \qquad \text{donc} \qquad \{\gamma^0, \gamma^0\} = 2 \cdot 1_{4\times 4},$$

et

$$\{\gamma^0, \gamma^i\} = \{\beta, \beta\alpha^i\} = \beta\beta\alpha^i + \beta\alpha^i\beta = \alpha^i + \beta\alpha^i\beta.$$

or

$$\begin{aligned} \beta\alpha^i\beta &= \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{0} & -\sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0} & -\sigma_i \\ -\sigma_i & \mathbf{0} \end{pmatrix} = -\alpha_i. \end{aligned}$$

Ainsi,

$$\{\gamma^0, \gamma^i\} = 0.$$

De plus, on a

$$\begin{aligned} \{\gamma^i, \gamma^j\} = \{\beta\alpha^i, \beta\alpha^j\} &= \beta\alpha^i\beta\alpha^j + \beta\alpha^j\beta\alpha^i \\ &= (-\alpha^i)\alpha^j + (-\alpha^j)\alpha^i \\ &= -\alpha^i\alpha^j - \alpha^j\alpha^i \\ &= -\{\alpha^i, \alpha^j\} \\ &= -2\delta_{ij}1_{4\times 4}. \end{aligned}$$

Avec $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, on a

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}1_{4\times 4}.$$

L'équation de Dirac s'écrit

$$i\partial_t\psi + i\alpha \cdot \nabla\psi - \beta m\psi = 0.$$

En multipliant cette équation Ã gauche par β, on obtient

$$i\beta\partial_t\psi + i\beta\alpha \cdot \nabla\psi - m\psi = 0$$

et avec les définitions des matrices de Dirac, on obtient

$$i\gamma^0\partial_t\psi + i\gamma^j\partial_j\psi - m\psi = 0$$

que l'on note

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0.}$$

b) Posons $\overline{\psi} = \psi^\dagger\gamma^0$. L'équation de Dirac conjuguée s'écrit

$$-i\partial_\mu\psi^\dagger(\gamma^\mu)^\dagger - m\psi^\dagger = 0$$

ou encore

$$-i\partial_0\psi^\dagger\gamma^0 + i\partial_j\psi^\dagger\gamma^j - m\psi^\dagger = 0,$$

où l'on a utilisé la relation $(\gamma^j)^\dagger = -\gamma^j$ et $(\gamma^0)^\dagger = \gamma^0$. En multipliant à droite par γ^0 et en utilisant l'identité $(\gamma^0)^2 = 1_{4\times4}$ on trouve

$$-i\partial_0(\psi^\dagger\gamma^0)\gamma^0 + i\partial_j\psi^\dagger(\gamma^0)^2\gamma^j\gamma^0 - m\psi^\dagger\gamma^0 = 0$$

soit

$$-i\partial_0\overline{\psi}\gamma^0 + i\partial_j\overline{\psi}\gamma^0\gamma^j\gamma^0 - m\overline{\psi} = 0.$$

Or on a

$$\gamma^0\gamma^j\gamma^0 = -\gamma^j\gamma^0\gamma^0 = -\gamma^j,$$

De sorte que l'équation de Dirac de $\overline{\psi}$ devient

$$-i\partial_t\overline{\psi}\gamma^0 - i\partial_j\overline{\psi}\gamma^j - m\overline{\psi} = 0.$$

soit

$$\boxed{-i\partial_\mu\overline{\psi}\gamma^\mu - m\overline{\psi} = 0.}$$

c) Avec le courant de Dirac

$$j^\mu = \overline{\psi}\gamma^\mu\psi,$$

on a

$$\begin{aligned}\partial_\mu j^\mu &= (\partial_\mu\overline{\psi}\gamma^\mu)\psi + \overline{\psi}\gamma^\mu\partial_\mu\psi \\ &= im\overline{\psi}\psi + \overline{\psi}(-im\psi) \\ &= im(\overline{\psi}\psi - \overline{\psi}\psi) \\ &= 0\end{aligned}$$

3.7.5 Problem 3.5 p.82

Soit les spineurs d'énergie positive

$$u(p,s) = (E+m)^{\frac{1}{2}}\begin{pmatrix}\phi^s \\ \frac{\sigma\cdot\mathbf{p}}{E+m}\phi^s\end{pmatrix} \qquad s = 1,2,$$

et d'énergie négative

$$v(p,s) = (E+m)^{\frac{1}{2}}\begin{pmatrix}\frac{\sigma\cdot\mathbf{p}}{E+m}\chi^s \\ \chi^s\end{pmatrix} \qquad s = 1,2.$$

On a alors

$$\begin{aligned}u^\dagger u &= (E+m)\begin{pmatrix}\phi^\dagger & \frac{\sigma\cdot\mathbf{p}}{E+m}\phi^\dagger\end{pmatrix}\begin{pmatrix}\phi \\ \frac{\sigma\cdot\mathbf{p}}{E+m}\phi\end{pmatrix} \\ &= (E+m)\left(\phi^\dagger\phi(1 + \frac{(\sigma\cdot\mathbf{p})(\sigma\cdot\mathbf{p})}{(E+m)^2})\right) \\ &= (E+m)\phi^\dagger\phi(\frac{(E+m)^2 + \mathbf{p}^2}{(E+m)^2}) \\ &= \phi^\dagger\phi(\frac{E^2 + 2mE + m^2 + E^2 - m^2}{E+m}) \\ &= \phi^\dagger\phi(\frac{2E^2 + 2Em}{E+m}) \\ &= 2E\phi^\dagger\phi.\end{aligned}$$

Comme les spineurs Ã deux composantes sont normalisés à 1, on a $\phi^\dagger\phi = 1$ et

$$u^\dagger u = 2E.$$

On a de même

$$\begin{aligned} v^\dagger v &= (E+m)\begin{pmatrix}\frac{\sigma\cdot\mathbf{p}}{E+m}\chi^\dagger & \chi^\dagger\end{pmatrix}\begin{pmatrix}\frac{\sigma\cdot\mathbf{p}}{E+m}\chi \\ \chi\end{pmatrix} \\ &= (E+m)\chi^\dagger\chi\left(\frac{(\sigma\cdot\mathbf{p})(\sigma\cdot\mathbf{p})}{(E+m)^2}+1\right) \\ &= (E+m)\left(\frac{\mathbf{p}^2}{(E+m)^2}+1\right) \\ &= \frac{E^2-m^2+E^2+2mE+m^2}{E+m} \\ &= \frac{2E^2+2Em}{E+m} \\ &= 2E. \end{aligned}$$

3.7.6 Problem 3.6 p.83

En définissant la dérivée covariante $D_\mu = \partial_\mu + iqA_\mu$ et en remplaçant ∂_μ par D_μ dans l'équation de Klein-Gordon, on obtient

$$(D_\mu D^\mu + m^2)\phi = 0.$$

En développant la dérivée covariante, on obtient

$$\begin{aligned} (D_\mu D^\mu + m^2)\phi &= ((\partial_\mu + iqA_\mu)(\partial^\mu + iqA^\mu) + m^2)\phi \\ &= ((\partial_\mu\partial^\mu + iq\partial_\mu A^\mu) + iqA_\mu(\partial^\mu + iqA^\mu) + m^2)\phi \\ &= (\partial_\mu\partial^\mu + iq(\partial_\mu A^\mu + A_\mu\partial^\mu) - q^2A_\mu A^\mu + m^2)\phi \\ &= 0 \end{aligned}$$

Ainsi, en isolant Ã gauche le terme de Klein-Gordon et en déplaçant les autres termes à droite, on obtient

$$(\Box + m^2)\phi = q^2A_\mu A^\mu\phi - iq(\partial_\mu A^\mu + A_\mu\partial^\mu)\phi.$$

On peut maintenant, en identifiant le membre de droite avec $-\hat{V}_{KG}\phi$ déduire l'expression de $\hat{V}_{KG}$

$$\hat{V}_{KG} = -q^2A_\mu A^\mu + iq(\partial_\mu A^\mu + A_\mu\partial^\mu).$$

3.8 Problems from the New edition 2002

3.8.1 Problem 4.2 p.101

3.8.2 Problem 4.5 p.102

3.8.3 Problem 4.6 p.103

3.8.4 Problem 4.7 p.103

3.8.5 Problem 4.8 p.103

3.8.6 Problem 4.9 p.103

3.8.7 Problem 4.10 p.103

3.8.8 Problem 4.14 p.104

3.8.9 Problem 4.15 p.104

Chapitre 4

Quantum Field Theory

4.1 Problems

4.1.1 Problem 4.1 p.125

Avec l'équation (4.33)

$$\phi(x,t) = \sum_{r=1}^{\infty} A_r(t) \sin\left(\frac{r\pi x}{L}\right),$$

on remplace dans l'expression de l'énergie (4.34)

$$E = \int_0^L \left[\frac{1}{2}\rho\big(\partial_t\phi\big)^2 + \frac{1}{2}\rho c^2\big(\partial_x\phi\big)^2\right] dx.$$

On a

$$\big(\partial_t\phi(x,t)\big)^2 = \sum_{r_1,r_2=1}^{\infty} \dot{A}_{r_1}\dot{A}_{r_2} \sin\left(\frac{r_1\pi x}{L}\right)\sin\left(\frac{r_2\pi x}{L}\right)$$

et

$$\big(\partial_x\phi(x,t)\big)^2 = \sum_{r_1,r_2=1}^{\infty} \left(\frac{r_2 r_1\pi^2}{L^2}\right) A_{r_1}A_{r_2} \cos\left(\frac{r_1\pi x}{L}\right)\cos\left(\frac{r_2\pi x}{L}\right).$$

L'intégrale sur x de 0 à L des sin et des cos dans chacune des sommes ci-dessus donne $\frac{L}{2}\delta_{r_1r_2}$, de sorte que l'expression de l'énergie devient

$$E = \frac{L}{2}\sum_{r=1}^{\infty} \frac{1}{2}\rho\dot{A}_r^2 + \frac{1}{2}\rho\omega_r^2 A_r^2$$

où $\omega_r = \frac{r^2\pi^2c^2}{L^2}$.

4.1.2 Problem 4.2 p.125

Soit le potentiel gravifique $V(x) = -mgx$, Le calcul de l'action sur un chemin $q(t)$ est donné par

$$S = \int_{t_1}^{t_2} dt\ L(q,\dot{q},t),$$

avec $t_1 = 0$ et $t_2 = t_0$ et où

$$L(q,\dot{q},t) = T - V = \frac{1}{2}m\dot{x}^2 + mgx,$$

pour les trajectoires suivantes

1. $x(t) = at$.
   On a alors
   $$T = \frac{1}{2}ma^2 \qquad \text{et} \qquad V = -mgat,$$
   de sorte que
   $$S_1(t_0) = \int_0^{t_0} \frac{1}{2}ma^2 + mgat\ dt = \frac{1}{2}ma^2t_0 + \frac{1}{2}mgat_0^2 = \frac{1}{2}mat_0(a+gt_0).$$

2. $x(t) = \frac{1}{2}gt^2$.
   On a alors
   $$T = \frac{1}{2}m(gt)^2 \qquad \text{et} \qquad V = -\frac{1}{2}mg(gt^2) = -\frac{1}{2}mg^2t^2$$
   et
   $$S_2(t_0) = \int_0^{t_0} \frac{1}{2}mg^2t^2 + \frac{1}{2}mg^2t^2 \; dt = \frac{1}{3}mg^2t_0^3.$$
3. $x(t) = bt^3$.
   On a alors
   $$T = \frac{9}{2}m(bt^2)^2 \qquad \text{et} \qquad V = -mgbt^3$$
   et
   $$S_3(t_0) = \int_0^{t_0} \frac{9}{2}mb^2t^4 + mgbt^3 \; dt = \frac{9}{5}mb^2t_0^5 + \frac{1}{4}mgbt_0^4.$$

Avec $x(t_0) = at_0 = \frac{1}{2}gt_0^2 = bt_0^3$, on a
$$a = \frac{1}{2}gt_0 \qquad \text{et} \qquad b = \frac{1}{2}\frac{g}{t_0}.$$
Ainsi
$$S_1 = \frac{1}{2}mat_0(a + gt_0) = \frac{1}{4}mgt_0^2(\frac{3}{2}gt_0) = \frac{3}{8}mg^2t_0^3.$$
et
$$S_3 = \frac{9}{5}m\frac{1}{4}g^2t_0^3 + \frac{1}{4}mg(\frac{1}{2}gt_0^2)t_0 = (\frac{9}{20} + \frac{1}{8})mg^2t_0^3.$$
En résumé
$$\begin{cases} S_1 &= \frac{45}{120}mg^2t_0^3 \\ S_2 &= \frac{40}{120}mg^2t_0^3 \\ S_3 &= \frac{69}{120}mg^2t_0^3. \end{cases}$$
Ainsi, on voit que
$$S_1 > S_2 \qquad \text{et} \qquad S_3 > S_2.$$

4.1.3 Problem 4.3 p.126

1. L'évolution de l'opérateur $\hat{q}$ est donnée par l'équation de Heisenberg
   $$\dot{\hat{q}} = -i[q, H]$$
   (avec $\hbar = 1$). Ainsi, pour l'hamiltonien (4.54,p98)
   $$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2,$$
   on a
   $$\begin{aligned} \dot{\hat{q}} &= -i[\hat{q}, \hat{H}] \\ &= -i[\hat{q}, \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2] \\ &= -\frac{i}{2}\left(\hat{q}(\frac{1}{m}\hat{p}^2 + m\omega^2\hat{q}^2) - (\frac{1}{m}\hat{p}^2 + m\omega^2\hat{q}^2)\hat{q}\right) \\ &= -\frac{i}{2}\left(\frac{1}{m}(\hat{p}\hat{q} + i)\hat{p} - \frac{1}{m}\hat{p}(\hat{q}\hat{p} - i)\right) \\ &= -\frac{i}{2m}2i\hat{p}. \end{aligned}$$
   On conclut, ainsi, que
   $$\boxed{m\dot{\hat{q}} = \hat{p}.}$$

2. De la même manière, on a

$$\begin{aligned}\dot{\hat{p}} &= -i[\hat{p}, \hat{H}] \\ &= -i[\hat{p}, \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2] \\ &= -\frac{i}{2}\left(\hat{p}\big(\frac{1}{m}\hat{p}^2 + m\omega^2\hat{q}^2\big) - \big(\frac{1}{m}\hat{p}^2 + m\omega^2\hat{q}^2\big)\hat{p}\right) \\ &= -\frac{i}{2}m\omega^2\left((\hat{q}\hat{p} - i)\hat{q} - \hat{q}(\hat{p}\hat{q} + i)\right) \\ &= -\frac{i}{2m}(-2i)\hat{q}.\end{aligned}$$

Ainsi,

$$\boxed{\dot{\hat{p}} = -m\omega^2\hat{q}.}$$

4.1.4 Problem 4.4 p.126

a) L'hamiltonien est donné par

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2.$$

Les opérateurs de création et d'annihilation sont donnés par

$$\begin{aligned}\hat{a} &= \frac{1}{\sqrt{2}}\big(\sqrt{m\omega}\hat{q} + \frac{i}{\sqrt{m\omega}}\hat{p}\big) \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2}}\big(\sqrt{m\omega}\hat{q} - \frac{i}{\sqrt{m\omega}}\hat{p}\big).\end{aligned}$$

On en déduit qu'en terme de ces opérateurs on a

$$\begin{aligned}\hat{q} &= \frac{1}{\sqrt{2m\omega}}(\hat{a} + \hat{a}^\dagger) \\ \hat{p} &= -i\sqrt{\frac{m\omega}{2}}(\hat{a} - \hat{a}^\dagger).\end{aligned}$$

Á partir de la relation de commutation $[q, p] = i$ on trouve

$$\begin{aligned}[\hat{a}, \hat{a}^\dagger] &= \frac{1}{2}\Big[\big(\sqrt{m\omega}\hat{q} + \frac{i}{\sqrt{m\omega}}\big)\big(\sqrt{m\omega}\hat{q} - \frac{i}{\sqrt{m\omega}}\hat{p}\big) \\ &\qquad - \big(\sqrt{m\omega}\hat{q} - \frac{i}{\sqrt{m\omega}}\hat{p}\big)\big(\sqrt{m\omega}\hat{q} + \frac{i}{\sqrt{m\omega}}\big) \\ &= \frac{1}{2}\big(i\hat{p}\hat{q} - i\hat{q}\hat{p} - i\hat{q}\hat{p} + i\hat{p}\hat{q}\big) \\ &= \frac{i}{2}2[\hat{p}, \hat{q}] \\ &= i(-i) = 1\end{aligned}$$

En remplaçant dans l'expression de l'hamiltonien les $\hat{p}$ et les $\hat{q}$ par les opérateurs de création et d'annihilation, on obtient

$$\begin{aligned}\hat{H} &= \frac{1}{2m}(-1)\frac{m\omega}{2}(\hat{a} - \hat{a}^\dagger)^2 + \frac{1}{2}m\omega^2\frac{1}{2m\omega}(\hat{a} + \hat{a}^\dagger)^2 \\ &= -\frac{\omega}{4}(\hat{a} - \hat{a}^\dagger)(\hat{a} - \hat{a}^\dagger) + \frac{\omega}{4}(\hat{a} + \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger) \\ &= \frac{\omega}{4}\big(-\hat{a}^2 - (\hat{a}^\dagger)^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^2 + (\hat{a}^\dagger)^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\big) \\ &= \frac{\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})\end{aligned}$$

en utilisant la relation de commutation ci-dessus, on trouve

$$\hat{H} = \big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\big)\omega.$$

b) On peut calculer les relations de commutations des opérateurs avec l'hamiltonien

$$\begin{aligned}[\hat{H}, \hat{a}] &= \omega[\hat{a}^\dagger \hat{a} + \frac{1}{2}, \hat{a}] = \omega[\hat{a}^\dagger \hat{a}, \hat{a}] \\ &= \omega(\hat{a}^\dagger \hat{a}^2 - \hat{a}\hat{a}^\dagger \hat{a}) = \omega\big(\hat{a}^\dagger \hat{a}^2 - (\hat{a}^\dagger \hat{a} + 1)\hat{a}\big) \\ &= -\omega \hat{a}\end{aligned}$$

et

$$\begin{aligned}[\hat{H}, \hat{a}^\dagger] &= \omega[\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] \\ &= \omega(\hat{a}^\dagger \hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a}^\dagger \hat{a}) = \omega\big(\hat{a}^\dagger(\hat{a}^\dagger \hat{a} + 1) - \hat{a}^\dagger \hat{a}^\dagger \hat{a}\big) \\ &= \omega \hat{a}^\dagger\end{aligned}$$

c) Pour vérifier la normalisation de

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle$$

commençons par calculer

$$\begin{aligned}\hat{a}^n(\hat{a}^\dagger)^n |0\rangle &= \hat{a}^{n-1}(\hat{a}^\dagger \hat{a} + 1)(\hat{a}^\dagger)^{n-1} |0\rangle \\ &= \hat{a}^{n-1}\hat{a}^\dagger \hat{a}(\hat{a}^\dagger)^{n-1} |0\rangle + \hat{a}^{n-1}(\hat{a}^\dagger)^{n-1} |0\rangle \\ &= \hat{a}^{n-1}(\hat{a}^\dagger)^k \hat{a}(\hat{a}^\dagger)^{n-k} |0\rangle + k\hat{a}^{n-1}(\hat{a}^\dagger)^{n-1} |0\rangle\end{aligned}$$

et pour $k = n$ on obtient

$$\hat{a}^n(\hat{a}^\dagger)^n |0\rangle = \hat{a}^{n-1}(\hat{a}^\dagger)^n \hat{a} |0\rangle + n\hat{a}^{n-1}(\hat{a}^\dagger)^{n-1} |0\rangle ,$$

où le premier terme est nul, car $\hat{a}|0\rangle = 0$. Il ne reste plus que

$$\hat{a}^n(\hat{a}^\dagger)^n |0\rangle = n\hat{a}^{n-1}(\hat{a}^\dagger)^{n-1} |0\rangle .$$

On voit tout de suite qu'en procédant de même pour les opérateur d'annihilation restants on obtient

$$\hat{a}^n(\hat{a}^\dagger)^n |0\rangle = n(n-1)\hat{a}^{n-2}(\hat{a}^\dagger)^{n-2} |0\rangle = n! |0\rangle .$$

Ainsi,

$$\langle n| n\rangle = \frac{1}{n!} \langle 0| \big(\hat{a}^n(\hat{a}^\dagger)^n |0\rangle\big) = \frac{n!}{n!} \langle 0| 0\rangle .$$

Ainsi, si $\langle 0| 0\rangle = 1$, on a bien

$$\langle n| n\rangle = 1.$$

d) Avec le résultat obtenu dans le point précédent, on peut calculer, en faisant passer l'opérateur d'annihilation vers la droite, à travers les n opérateurs de création,

$$\begin{aligned}\hat{a}^\dagger \hat{a} |n\rangle &= \frac{1}{\sqrt{n!}}\hat{a}^\dagger \hat{a}(\hat{a}^\dagger)^n |0\rangle \\ &= \frac{1}{\sqrt{n!}}\hat{a}^\dagger n(\hat{a}^\dagger)^{n-1} |0\rangle \\ &= n\frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n |0\rangle \\ &= n |n\rangle .\end{aligned}$$

Ainsi, on a

$$\hat{n} |n\rangle = \hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle .$$

4.1.5 Problem 4.5 p.126

En partant de la densité lagrangienne

$$\mathcal{L} = i\psi^* \dot{\psi} - \frac{1}{2m}\nabla\psi^* \cdot \nabla\psi$$

les équations d'Euler-Lagrange par rapport à ψ donnent

$$\frac{\delta \mathcal{L}}{\delta \psi} = \partial_t \left(\frac{\delta \mathcal{L}}{\delta \dot{\psi}} \right) + \nabla \left(\frac{\delta \mathcal{L}}{\delta (\nabla \psi)} \right)$$

soit

$$0 = i\dot{\psi}^* + \frac{-1}{2m} \nabla(\nabla \psi^*)$$

d'où l'équation de Schrœdinger pour ψ^*

$$\boxed{i\partial \psi^* = \frac{1}{2m} \nabla^2 \psi^*}.$$

L'équation d'Euler-Lagrange pour ψ^* donne

$$i\partial_t \psi = 0 - \frac{1}{2m} \nabla(\nabla \psi)$$

d'où l'équation de Schrœdinger pour ψ

$$\boxed{i\partial_t \psi = -\frac{1}{2m} \nabla^2 \psi.}$$

On constate que ces deux équations sont bien conjuguées l'une de l'autre.

Dans les exercices suivants on ne notera plus le chapeau sur les lettres pour désigner des opérateurs, leur utilisation rend la nature des opérateurs claire.

4.1.6 Problem 4.6 p.126

Les relations de commutation des opérateurs de création et d'annihilation sont données par la relation (4.104)

$$[a_k, a_{k'}^\dagger] = 2\pi 2\omega \delta(k-k'), \qquad \text{et} \qquad [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0.$$

Pour la relation de commutation de ϕ et π on obtient

$$\begin{aligned}
[\phi(x,t), \pi(y,t)] &= \int \frac{dk\, dk'}{(2\pi)^2} \frac{1}{2\omega} \frac{-i\omega'}{2\omega'} \\
&\quad [a_k e^{ikx - i\omega t} + a_k^\dagger e^{-ikx+i\omega t}, a_{k'} e^{ik'y - i\omega' t} - a_{k'}^\dagger e^{-ik'y + i\omega' t}] \\
&= \frac{-i}{2} \int \frac{dk\, dk'}{(2\pi)^2} \frac{1}{2\omega} [a_k, a_{k'}] e^{i(kx+k'y) - i(\omega+\omega')t} \\
&\qquad - [a_k^\dagger, a_{k'}^\dagger] e^{-i(kx+k'y)+i(\omega+\omega')t} \\
&\qquad - [a_k, a_{k'}^\dagger] e^{i(kx-k'y)-i(\omega-\omega')t} \\
&\qquad + [a_k^\dagger, a_{k'}] e^{-i(kx-k'y)+i(\omega-\omega')t} \\
&= \frac{-i}{2} \int \frac{dk\, dk'}{(2\pi)^2} \frac{1}{2\omega} \Big(- 2\pi 2\omega \delta(k-k') e^{i(kx-k'y)-i(\omega-\omega')t} \\
&\qquad - 2\pi 2\omega \delta(k-k') e^{-i(kx-k'y)+i(\omega-\omega')t} \Big) \\
&= \frac{i}{2} \int \frac{dk}{2\pi} e^{ik(x-y)} + e^{-ik(x-y)} = \frac{i}{2} \big(2\delta(x-y) \big)
\end{aligned}$$

Ainsi, on obtient bien la relation (4.95)

$$\boxed{[\phi(x,t), \pi(y,t)] = i\delta(x-y).}$$

4.1.7 Problem 4.7 p.126

La densité hamiltonienne de la corde vibrante est donné par

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x\phi)^2,$$

avec

$$\pi = \partial_t\phi \qquad \text{et} \qquad H = \int dx\ \mathcal{H},$$

où H est l'opérateur hamiltonien.

On a de plus

$$\phi(x,t) = \int \frac{dk}{2\pi}\frac{1}{2\omega}(a_k e^{ikx-i\omega t} + a_k^\dagger e^{-ikx+i\omega t})$$

et

$$\pi(x,t) = \partial_t\phi = \int \frac{dk}{2\pi}\frac{1}{2\omega}(-i\omega)(a_k e^{ikx-i\omega t} - a_k^\dagger e^{-ikx+i\omega t}).$$

On peut alors écrire

$$\begin{aligned}\pi(x,t)^2 = &\int \frac{dk\,dk'}{(2\pi)^2}\frac{-i\omega}{2\omega}\frac{-i\omega'}{2\omega'} \\ &\qquad (a_k e^{ikx-i\omega t} - a_k^\dagger e^{-ikx+i\omega t})(a_{k'} e^{ik'x-i\omega' t} - a_{k'}^\dagger e^{-ik'x+i\omega' t}) \\ = &-\frac{1}{4}\int \frac{dk\,dk'}{(2\pi)^2}\ a_k a_{k'} e^{i(k+k')x-i(\omega+\omega')t} + a_k^\dagger a_{k'}^\dagger e^{-i(k+k')x+i(\omega+\omega')t} \\ &\qquad - a_k a_{k'}^\dagger e^{i(k-k')x-i(\omega-\omega')t} - a_k^\dagger a_{k'} e^{-i(k-k')x+i(\omega-\omega')t}\end{aligned}$$

et avec

$$\partial_x\phi = \int \frac{dk}{2\pi}\frac{1}{2\omega}(ik)(a_k e^{ikx-i\omega t} - a_k^\dagger e^{-ikx+i\omega t})$$

on a

$$\begin{aligned}\partial\phi(x,t)^2 = &\int \frac{dk\,dk'}{(2\pi)^2}\frac{ik}{2\omega}\frac{ik'}{2\omega'} \\ &\qquad (a_k e^{ikx-i\omega t} - a_k^\dagger e^{-ikx+i\omega t})(a_{k'} e^{ik'x-i\omega' t} - a_{k'}^\dagger e^{-ik'x+i\omega' t}) \\ = &-\frac{1}{4}\int \frac{dk\,dk'}{(2\pi)^2}\frac{kk'}{|k|\,|k'|}\ a_k a_{k'} e^{i(k+k')x-i(\omega+\omega')t} + a_k^\dagger a_{k'}^\dagger e^{-i(k+k')x+i(\omega+\omega')t} \\ &\qquad - a_k a_{k'}^\dagger e^{i(k-k')x-i(\omega-\omega')t} - a_k^\dagger a_{k'} e^{-i(k-k')x+i(\omega-\omega')t}.\end{aligned}$$

Ainsi

$$\begin{aligned}\mathcal{H} = \frac{1}{2}\frac{-1}{4}\int \frac{dk\,dk'}{(2\pi)^2}\left(1 + \frac{kk'}{|k|\,|k'|}\right)&\big(a_k a_{k'} e^{i(k+k')x-i(\omega+\omega')t} + a_k^\dagger a_{k'}^\dagger e^{-i(k+k')x+i(\omega+\omega')t} \\ &- a_k a_{k'}^\dagger e^{i(k-k')x-i(\omega-\omega')t} - a_k^\dagger a_{k'} e^{-i(k-k')x+i(\omega-\omega')t}\big)\end{aligned}$$

L'hamiltonien s'obtient en intégrant sur x

$$\begin{aligned}H &= \int dx\ \mathcal{H} \\ &= \frac{-1}{8}\int \frac{dk\,dk'}{2\pi}\left(1 + \frac{kk'}{|k|\,|k'|}\right)\big(a_k a_{k'}\delta(k+k')e^{-i(\omega+\omega')t} \\ &\qquad + a_k^\dagger a_{k'}^\dagger \delta(k+k')e^{i(\omega+\omega')t} - a_k a_{k'}^\dagger \delta(k-k')e^{-i(\omega-\omega')t} \\ &\qquad - a_k^\dagger a_{k'}\delta(k-k')e^{i(\omega-\omega')t}\big) \\ &= \frac{-1}{8}\int \frac{dk}{2\pi}\left(1 + \frac{k(-k)}{|k|^2}\right)\big(a_k a_{-k} e^{-i(2\omega)t} + a_k^\dagger a_{-k}^\dagger e^{i(2\omega)t}\big) \\ &\qquad - \left(1 + \frac{k(+k)}{|k|^2}\right)\big(a_k a_k^\dagger + a_k^\dagger a_k\big) \\ &= \frac{1}{8}\int \frac{dk}{2\pi} 2\big(a_k a_k^\dagger + a_k^\dagger a_k\big) \\ &= \int \frac{dk}{2\pi}\frac{1}{2\omega}\left[\frac{1}{2}\big(a_k a_k^\dagger + a_k^\dagger a_k\big)\right]\omega.\end{aligned}$$

c'est la formule (4.106)

$$\boxed{H = \int \frac{dk}{2\pi}\frac{1}{2\omega}\left[\frac{1}{2}\left(a_k^\dagger a_k + a_k a_k^\dagger\right)\right]\omega.}$$

4.1.8 Problem 4.8 p.126

Soit la densité lagrangienne de deux champs réels scalaires ϕ_1 et ϕ_2

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2\phi_2^2.$$

On opère une rotation dans tout l'espace (transformation de jauge de 1re espèce) des champs

$$\begin{cases}\phi_1' &= \cos(\alpha)\phi_1 - \sin(\alpha)\phi_2\\ \phi_2' &= \sin(\alpha)\phi_1 + \cos(\alpha)\phi_2\end{cases} \Longleftrightarrow \begin{cases}\phi_1 &= \cos(\alpha)\phi_1' + \sin(\alpha)\phi_2'\\ \phi_2 &= -\sin(\alpha)\phi_1' + \cos(\alpha)\phi_2'.\end{cases}$$

On a alors

$$\begin{cases}\partial_\mu\phi_1 &= \cos(\alpha)\partial_\mu\phi_1' + \sin(\alpha)\partial_\mu\phi_2'\\ \partial_\mu\phi_2 &= -\sin(\alpha)\partial_\mu\phi_1' + \cos(\alpha)\partial_\mu\phi_2'.\end{cases}$$

Ainsi, la densité lagrangienne s'écrit

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}(\cos(\alpha)\partial_\mu\phi_1' + \sin(\alpha)\partial_\mu\phi_2')(\cos(\alpha)\partial^\mu\phi_1' + \sin(\alpha)\partial^\mu\phi_2')\\ &- \frac{1}{2}(\cos(\alpha)\phi_1' + \sin(\alpha)\phi_2')(\cos(\alpha)\phi_1' + \sin(\alpha)\phi_2')\\ &+ \frac{1}{2}(-\sin(\alpha)\partial_\mu\phi_1' + \cos(\alpha)\partial_\mu\phi_2')(-\sin(\alpha)\partial^\mu\phi_1' + \cos(\alpha)\partial^\mu\phi_2')\\ &- \frac{1}{2}(-\sin(\alpha)\phi_1' + \cos(\alpha)\phi_2')(-\sin(\alpha)\phi_1' + \cos(\alpha)\phi_2')\end{aligned}$$

Ce qui donne

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\cos^2(\alpha)\partial_\mu\phi_1'\partial^\mu\phi_1' + \sin^2(\alpha)\partial_\mu\phi_2'\partial^\mu\phi_2' + 2\sin(\alpha)\cos(\alpha)\partial_\mu\phi_1'\partial^\mu\phi_2')\\ &\quad - \frac{1}{2}(\cos^2(\alpha)(\phi_1')^2 + \sin^2(\alpha)(\phi_2')^2 + 2\sin(\alpha)\cos(\alpha)\phi_1'\phi_2')\\ &\quad + \frac{1}{2}(\sin^2(\alpha)\partial_\mu\phi_1'\partial^\mu\phi_1' + \cos^2(\alpha)\partial_\mu\phi_2'\partial^\mu\phi_2' - 2\sin(\alpha)\cos(\alpha)\partial_\mu\phi_1'\partial^\mu\phi_2')\\ &\quad - \frac{1}{2}(\sin^2(\alpha)(\phi_1')^2 + \cos^2(\alpha)(\phi_2')^2 - 2\sin(\alpha)\cos(\alpha)\phi_1'\phi_2')\\ &= \frac{1}{2}\partial_\mu\phi_1'\partial^\mu\phi_1' + \frac{1}{2}\partial_\mu\phi_2'\partial^\mu\phi_2' - \frac{1}{2}m^2\phi_1'^2 - \frac{1}{2}m^2\phi_2'^2\\ &= \frac{1}{2}\partial_\mu\phi_1'\partial^\mu\phi_1' - \frac{1}{2}m^2\phi_1'^2 + \frac{1}{2}\partial_\mu\phi_2'\partial^\mu\phi_2' - \frac{1}{2}m^2\phi_2'^2.\end{aligned}$$

Ainsi, la densité lagrangienne est invariante sous les transformations de jauge de 1re espèce.

4.1.9 Problem 4.9 p.126

a) l'opérateur N^μ est défini par la relation (4.158)

$$N^\mu = i(\phi^\dagger\partial^\mu\phi - \phi\partial^\mu\phi^\dagger)$$

et l'opérateur *charge conservée* est donné par la relation (4.149)

$$N = \int N^0\ d^3x.$$

Avec les champs

$$\phi = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}(a_k e^{-ik\cdot r} + b_k^\dagger e^{ik\cdot r})$$

et

$$\phi^\dagger = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} (b_k e^{-ik\cdot r} + a_k^\dagger e^{ik\cdot r})$$

on obtient, avec $k^0 = \omega$

$$\partial^0 \phi = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} (-i\omega)(a_k e^{-ik\cdot r} - b_k^\dagger e^{ik\cdot r})$$

et

$$\partial^0 \phi^\dagger = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} (-i\omega)(b_k e^{-ik\cdot r} - a_k^\dagger e^{ik\cdot r}).$$

Ainsi

$$\begin{aligned} N^0 &= i(\phi^\dagger \partial^0 \phi - \phi \partial^0 \phi^\dagger) \\ &= \int \frac{d^3k\, d^3k'}{(2\pi)^6} \frac{1}{2\omega} \frac{1}{2} \Big[(b_k e^{-ik\cdot r} + a_k^\dagger e^{ik\cdot r})(a_{k'} e^{-ik'\cdot r} - b_{k'}^\dagger e^{ik'\cdot r}) \\ &\qquad - (a_k e^{-ik\cdot r} + b_k^\dagger e^{ik\cdot r})(b_{k'} e^{-ik'\cdot r} - a_{k'}^\dagger e^{ik'\cdot r}) \Big] \end{aligned}$$

soit

$$\begin{aligned} N^0 = \int \frac{d^3k\, d^3k'}{(2\pi)^6} \frac{1}{2\omega} \frac{1}{2} \Big[&\big(b_k a_{k'} e^{-i(k+k')\cdot r} + a_k^\dagger a_{k'} e^{i(k-k')\cdot r} \\ &- b_k b_{k'}^\dagger e^{-i(k-k')\cdot r} - a_k^\dagger b_{k'}^\dagger e^{i(k+k')\cdot r} \big) \\ &- \big(a_k b_{k'} e^{-i(k+k')\cdot r} + b_k^\dagger b_{k'} e^{i(k-k')\cdot r} \\ &- a_k a_{k'}^\dagger e^{-i(k-k')\cdot r} - b_k^\dagger a_{k'}^\dagger e^{i(k'+k)\cdot r} \big) \Big] \\ = \int \frac{d^3k\, d^3k'}{(2\pi)^6} \frac{1}{2\omega} \frac{1}{2} \Big[&(b_k a_{k'} - a_k b_{k'}) e^{-i(k+k')\cdot r} + (a_k^\dagger a_{k'} + a_k a_{k'}^\dagger) e^{i(k-k')\cdot r} \\ &- (b_k b_{k'}^\dagger + b_k^\dagger b_{k'}) e^{-i(k-k')\cdot r} - (b_k^\dagger a_{k'}^\dagger - a_k^\dagger b_{k'}^\dagger) e^{i(k+k')\cdot r} \Big] \end{aligned}$$

On passe à N en intégrant sur r

$$\begin{aligned} N &= \int d^3r \; N^0 \\ &= \int \frac{d^3k\, d^3k'}{(2\pi)^3} \frac{1}{2\omega} \frac{1}{2} \Big[(b_k a_{k'} - a_k b_{k'}) \delta(k+k') + (a_k^\dagger a_{k'} + a_k a_{k'}^\dagger) \delta(k-k') \\ &\qquad - (b_k b_{k'}^\dagger + b_k^\dagger b_{k'}) \delta(k-k') - (b_k^\dagger a_{k'}^\dagger - a_k^\dagger b_{k'}^\dagger) \delta(k+k') \Big] \end{aligned}$$

En renommant les variables d'intégration dans un des termes de chaque parenthèse, on fait apparaître les commutateurs des opérateurs a et b. Ils s'annulent tous, sauf ceux qui font intervenir a et $a^\dagger$ et ceux qui font intervenir b et $b^\dagger$. Ne restent, finalement, plus que les termes en $\delta(k-k')$, soit

$$\begin{aligned} N &= \int \frac{d^3k\, d^3k'}{(2\pi)^3} \frac{1}{2\omega} \frac{1}{2} \Big[(a_k^\dagger a_{k'} + a_{k'} a_k^\dagger) \delta(k-k') - (b_k b_{k'}^\dagger + b_{k'}^\dagger b_k) \delta(k-k') \Big] \\ &= \int \frac{d^3k\, d^3k'}{(2\pi)^3} \frac{1}{2\omega} \frac{1}{2} \Big[(a_k^\dagger a_{k'} + a_{k'} a_k^\dagger) - (b_k b_{k'}^\dagger + b_{k'}^\dagger b_k) \Big] \delta(k-k') \\ &= \int \frac{d^3k\, d^3k'}{(2\pi)^3} \frac{1}{2\omega} \frac{1}{2} \Big[(a_k^\dagger a_{k'} + a_k^\dagger a_{k'} + (2\pi)^3 2\omega \delta(k-k')) \\ &\qquad - (b_k^\dagger b_{k'} + b_k^\dagger b_{k'} + (2\pi)^3 2\omega \delta(k-k')) \Big] \delta(k-k') \\ &= \int \frac{d^3k\, d^3k'}{(2\pi)^3} \frac{1}{2\omega} \frac{1}{2} \Big[2a_k^\dagger a_{k'} - 2b_k^\dagger b_{k'} \Big] \delta(k-k') \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left[a_k^\dagger a_k - b_k^\dagger b_k \right] \end{aligned}$$

Ainsi, on obtient le résultat (4.159)

$$\boxed{N = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left[a_k^\dagger a_k - b_k^\dagger b_k\right].}$$

b) L'opérateur hamiltonien est donné par

$$H = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left[a_k^\dagger a_k + b_k^\dagger b_k\right] \omega.$$

Pour calculer commutateur de l'opérateur N avec l'hamiltonien H, il suffit de calculer le commutateur

$$[a_k^\dagger a_k - b_k^\dagger b_k, a_{k'}^\dagger a_{k'} + b_{k'}^\dagger b_{k'}] = [a_k^\dagger a_k, a_{k'}^\dagger a_{k'}] - [b_k^\dagger b_k, b_{k'}^\dagger b_{k'}].$$

les autres commutateurs étant nuls, puisque les a et les b commutent.
On a

$$\begin{aligned}
[a_k^\dagger a_k, a_{k'}^\dagger a_{k'}] &= \left(a_k^\dagger a_k a_{k'}^\dagger a_{k'} - a_{k'}^\dagger a_{k'} a_k^\dagger a_k\right) \\
&= \left(a_k^\dagger a_{k'}^\dagger a_k a_{k'} + a_k^\dagger a_{k'} \delta(k-k') - a_{k'}^\dagger a_k^\dagger a_{k'} a_k - a_{k'}^\dagger a_k \delta(k-k')\right) \\
&= 0
\end{aligned}$$

car avec la contrainte du $\delta(k - k')$, on a $k = k'$.
Il en va exactement de même pour le commutateur des b. On en conclut

$$[N, H] = 0,$$

ce qui montre que N est bien une quantité conservée.

4.1.10 Problem 4.10 p.127

La densité lagrangienne du champ de Dirac est donnée par la relation (4.165)

$$\mathcal{L}_D = i\psi^\dagger \dot{\psi} + i\psi^\dagger \alpha \cdot \nabla\psi - m\psi^\dagger \beta\psi.$$

La variation de $\mathcal{L}_D$ par rapport à $\psi^\dagger$ donne

$$\begin{aligned}
0 &= \partial_\mu \frac{\delta \mathcal{L}_D}{\delta(\partial^\mu \psi^\dagger)} - \frac{\delta \mathcal{L}_D}{\delta \psi^\dagger} \\
&= \partial_\mu(0) - i\dot{\psi} - i\alpha \cdot \nabla\psi + m\beta\psi
\end{aligned}$$

c'est-à -dire l'équation de Dirac

$$i\partial_t \psi + \alpha \cdot \nabla\psi - m\psi = 0.$$

En multipliant par β à droite avec la définition des matrices γ^μ on trouve

$$(i\gamma_\mu \partial^\mu - m)\psi = 0,$$

l'équation de Dirac sous sa forme manifestement covariante.

4.1.11 Problem 4.11 p.127

Soit la densité lagrangienne

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu,$$

avec $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. La variation de $F^{\mu\nu} F_{\mu\nu}$ par rapport à $\partial_\nu A_\mu$ donne

$$\begin{aligned}
0 &= \partial_\nu \frac{\delta \mathcal{L}_D}{\delta(\partial_\nu A_\mu)} \\
&= \partial^\mu A^\nu - \partial^\nu A^\mu \\
&= -F^{\nu\mu}
\end{aligned}$$

(cf. Peskin-Schroeder 2.1 à la page 100.) Ainsi, la première partie de l'équation d'Euler-Lagrange, s'écrit

$$\partial_\nu \frac{\delta \mathcal{L}_D}{\delta(\partial_\nu A_\mu)} = \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\nu(\partial^\mu A^\nu) - \Box A^\mu.$$

Le second terme vient de

$$\frac{\delta \mathcal{L}}{\delta A_\mu} = -j^\mu.$$

Les équation d'Euler-Lagrange donnent alors

$$\begin{aligned} 0 &= \partial_\nu \frac{\delta \mathcal{L}_D}{\delta(\partial_\nu A_\mu)} - \frac{\delta \mathcal{L}_D}{\delta A_\mu} \\ &= \partial_\nu(\partial^\mu A^\nu) - \Box A^\mu + j_\mu. \end{aligned}$$

On trouve ainsi

$$\boxed{\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu.}$$

Chapitre 5

QED pour particules de spins 0

5.1 Notes

5.1.1 États créés par les opérateurs p.138

En général, on a pour les états à un photon

$$|k\lambda\rangle \propto \alpha^{\dagger}_{k\lambda}|0\rangle$$

et pour les états à particules

$$|p\rangle \propto a^{\dagger}_{p}|0\rangle .$$

Avec un coefficient de normalisation N_k et N_p on peut alors écrire

$$|k\lambda\rangle = N_k \alpha^{\dagger}_{k\lambda}|0\rangle$$

et

$$|p\rangle = N_p a^{\dagger}_{p}|0\rangle .$$

Les éléments de matrices (5.43) et (5.44) sont ainsi donnés par

$$\langle 0 |A_\mu(x)| k\lambda\rangle = N_k \left\langle 0 \middle| A_\mu(x)\alpha^{\dagger}_{k\lambda} \middle| 0\right\rangle$$

et

$$\langle p_f |j^\mu(x)| p_i\rangle = N_{p_f}N_{p_i}\left\langle 0 \middle| a_{p_f} j^\mu(x) a^{\dagger}_{p_i} \middle| 0\right\rangle .$$

5.2 Problems

5.2.1 Problem 5.1 p.169

On considère un élément de matrice de la forme

$$M = \int d^3x \int dt\ e^{ip_f\cdot x}\partial_\mu A^\mu e^{-ip_i\cdot x}.$$

L'intégration par parties du terme temporel et du terme spatial donnent respectivement

$$\begin{aligned}\int dt\ e^{ip_f\cdot x}\partial_0 A^0 e^{-ip_i\cdot x} &= -\int dt\ \partial_0(e^{ip_f\cdot x})A^0 e^{-ip_i\cdot x} + \text{terme nul en } \infty \\ &= -ip_{f0}\int dt\ e^{ip_f\cdot x}A^0 e^{-ip_i\cdot x}\end{aligned}$$

et

$$\begin{aligned}\int dt\ e^{ip_f\cdot x}\nabla\cdot\mathbf{A}e^{-ip_i\cdot x} &= -\int dt\ \nabla(e^{ip_f\cdot x})\mathbf{A}e^{-ip_i\cdot x} + 0 \\ &= -(-i\mathbf{p}_f)\cdot\int dt\ e^{ip_f\cdot x}\mathbf{A}e^{-ip_i\cdot x} \\ &= i\mathbf{p}_f\cdot\int dt\ e^{ip_f\cdot x}\mathbf{A}e^{-ip_i\cdot x}\end{aligned}$$

Ainsi

$$M = \int d^3x \int dt\ e^{ip_f\cdot x}\partial_\mu A^\mu e^{-ip_i\cdot x} = -ip_{f\mu}\int dt\ e^{ip_f\cdot x}A^\mu e^{-ip_i\cdot x}.$$

On a alors

$$\begin{aligned}
&\int d^3x \int dt\ e^{ip_f\cdot x}(\partial_\mu A^\mu + A_\mu\partial^\mu)e^{-ip_i\cdot x}\\
&\quad= \int d^3x \int dt\ e^{ip_f\cdot x}\partial_\mu A^\mu e^{-ip_i\cdot x} + \int d^3x \int dt\ e^{ip_f\cdot x}A_\mu\partial^\mu e^{-ip_i\cdot x}\\
&\quad= -ip_{f\mu}\int d^4x\ e^{ip_f\cdot x}A^\mu e^{-ip_i\cdot x} - ip_{i\mu}\int d^4x\ e^{ip_f\cdot x}A^\mu e^{-ip_i\cdot x}\\
&\quad= -i(p_f+p_i)_\mu\int d^4x\ e^{ip_f\cdot x}A^\mu e^{-ip_i\cdot x}
\end{aligned}$$

5.2.2 Problem 5.2 p.170

Établissons la formule (5.59)

$$\left\langle p_f \left| \hat{j}^\mu_{\text{em}}(x) \right| p_i \right\rangle = eN_fN_i(p_i+p_f)^\mu e^{-i(p_i-p_f)\cdot x}.$$

La densité de courant du champs scalaire de Klein-Gordon s'écrit

$$\hat{j}^\mu_{\text{em}} = ie(\hat{\phi}^\dagger\partial^\mu\hat{\phi} - (\partial^\mu\hat{\phi}^\dagger)\hat{\phi})$$

avec

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3 2E}[\hat{a}_p e^{-ip\cdot x} + \hat{b}^\dagger_p e^{ip\cdot x}].$$

et

$$\partial^\mu\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3 2E}(-ip^\mu)[\hat{a}_p e^{-ip\cdot x} - \hat{b}^\dagger_p e^{ip\cdot x}].$$

et pour les opérateurs de champs conjugués

$$\hat{\phi}^\dagger(x) = \int \frac{d^3p}{(2\pi)^3 2E}[\hat{b}_p e^{-ip\cdot x} + \hat{a}^\dagger_p e^{ip\cdot x}].$$

et

$$\partial^\mu\hat{\phi}(x)^\dagger = \int \frac{d^3p}{(2\pi)^3 2E}(-ip^\mu)[\hat{b}_p e^{-ip\cdot x} - \hat{a}^\dagger_p e^{ip\cdot x}].$$

On développe j^μ (on laisse tomber le chapeau sur les opérateurs)

$$\begin{aligned}
j^\mu = ie(-i)\int \frac{d^3p\,d^3p'}{(2\pi)^6}\frac{1}{2E_p 2E_{p'}}\Big[&p'^\mu[b_p e^{-ip\cdot x} + a^\dagger_p e^{ip\cdot x}][a_{p'}e^{-ip'\cdot x} - b^\dagger_{p'}e^{ip'\cdot x}]\\
&- p^\mu[b_p e^{-ip\cdot x} - a^\dagger_p e^{ip\cdot x}][a_{p'}e^{-ip'\cdot x} + b^\dagger_{p'}e^{ip'\cdot x}]\Big]\\
= e\int \frac{d^3p\,d^3p'}{(2\pi)^6}\frac{1}{2E_p 2E_{p'}}\Big[&p'^\mu[b_p a_{p'}e^{-i(p+p')\cdot x} + a^\dagger_p a_{p'}e^{i(p-p')\cdot x} - b_p b^\dagger_{p'}e^{-i(p-p')\cdot x}\\
&- a^\dagger_p b^\dagger_{p'}e^{i(p+p')\cdot x}]\\
&- p^\mu[b_p a_{p'}e^{-i(p+p')\cdot x} - a^\dagger_p a_{p'}e^{i(p-p')\cdot x} + b_p b^\dagger_{p'}e^{-i(p-p')\cdot x}\\
&- a^\dagger_p b^\dagger_{p'}e^{i(p+p')\cdot x}]\Big]
\end{aligned}$$

On peut alors calculer $\langle p_f\,| j^\mu |\, p_i\rangle$ en évaluant

$$\langle p_f\,| j^\mu |\, p_i\rangle = N_{p_f}N_{p_i}\left\langle 0 \left| a_{p_f} j^\mu a^\dagger_{p_i} \right| 0\right\rangle.$$

On doit alors calculer

$$\left\langle 0 \left| a_{p_f} b_p a_{p'} a^\dagger_{p_i} \right| 0\right\rangle = \left\langle 0 \left| a_{p_f} a_{p'} b_p a^\dagger_{p_i} \right| 0\right\rangle = \left\langle 0 \left| a_{p_f} a_{p'} a^\dagger_{p_i} b_p \right| 0\right\rangle = 0$$

car les a et les b commutent.

$$\begin{aligned}\langle 0 \left| a_{p_f} a_p^\dagger a_{p'} a_{p_i}^\dagger \right| 0 \rangle &= \langle 0 \left| (a_p^\dagger a_{p_f} + (2\pi)^3 2E_p \delta(p_f - p))(a_{p_i}^\dagger a_{p'} + (2\pi)^3 2E_{p'} \delta(p_i - p')) \right| 0 \rangle \\ &= (2\pi)^3 2E_p (2\pi)^3 2E_{p'} \langle 0 \left| \delta(p_f - p)\delta(p_i - p') \right| 0 \rangle \\ &= (2\pi)^3 2E_p (2\pi)^3 2E_{p'} \delta(p_f - p)\delta_(p_i - p')\end{aligned}$$

$$\left\langle 0 \left| a_{p_f} b_p b_{p'}^\dagger a_{p_i}^\dagger \right| 0 \right\rangle = \left\langle 0 \left| b_p b_{p'}^\dagger a_{p_f} a_{p_i}^\dagger \right| 0 \right\rangle = \left\langle 0 \left| b_p b_{p'}^\dagger a_{p_f} a_{p_i}^\dagger \right| 0 \right\rangle = 0$$

si $p_i \neq p_f$. Enfin

$$\left\langle 0 \left| a_{p_f} a_p^\dagger b_{p'}^\dagger a_{p_i}^\dagger \right| 0 \right\rangle = \left\langle 0 \left| (a_p^\dagger a_{p_f} + \delta(p - p_f)) b_{p'}^\dagger a_{p_i}^\dagger \right| 0 \right\rangle = 0,$$

car $a^\dagger \left| 0 \right\rangle = b^\dagger \left| 0 \right\rangle = 0$. On obtient

$$\begin{aligned}\langle p_f \left| j^\mu \right| p_i \rangle = eN_{p_f} N_{p_i} \int \frac{d^3p \, d^3p'}{(2\pi)^6} \frac{1}{2E_p 2E_{p'}} \Big[& p'^\mu (2\pi)^3 2E_p (2\pi)^3 2E_{p'} \delta(p_f - p)\delta_(p_i - p') \\ & + p^\mu (2\pi)^3 2E_p (2\pi)^3 2E_{p'} \delta(p_f - p)\delta_(p_i - p') \Big] e^{i(p - p')\cdot x} \\ = eN_{p_f} N_{p_i} \int d^3p \, d^3p' \Big[p'^\mu + p^\mu \Big] & \delta(p_f - p)\delta_(p_i - p') e^{i(p - p')\cdot x} \\ = eN_{p_f} N_{p_i} (p_i + p_f)^\mu & e^{-i(p_i - p_f)\cdot x}\end{aligned}$$

soit

$$\boxed{\langle p_f \left| j^\mu \right| p_i \rangle = eN_{p_f} N_{p_i} (p_i + p_f)^\mu e^{-i(p_i - p_f)\cdot x}.}$$

C'est la relation (5.59) p.139.

5.2.3 Problem 5.3 p.170

Pour calculer l'amplitude de diffusion électromagnétique d'une particule b de charge e_b et de masse m_b par une antiparticule $\bar{a}$ d'une particule a de charge e_a et de masse m_a on prend en compte que $e_{\bar{a}} = -e_a$. Il faut calculer

$$\mathcal{A}_{fi} = -i \int d^4x \left\langle p_3 \left| j^\mu_{\bar{a}\bar{a}'}(x) \right| p_1 \right\rangle \langle 0 \left| A_\mu(x) \right| k\lambda \rangle$$

avec

$$\langle p_3 \left| j^\mu_{\bar{a}\bar{a}'}(x) \right| p_1 \rangle = N_1 N_3 \left\langle 0 \left| b_3 j^\mu_{\bar{a}\bar{a}'}(x) b_1^\dagger \right| 0 \right\rangle$$

où les opérateurs b et $b^\dagger$ sont les opérateurs d'annihilation et de création des antiparticules $\bar{a}$ et avec $A_\mu(x)$ le champ en x dû au courant chargé des particules b

$$A_\mu(x) = e_b N_2 N_4 (p_2 + p_4)_\mu \frac{-1}{q^2} e^{-i(p_4 - p_2)\cdot x}$$

où $q = p_4 - p_2$.

Pour évaluer $j^\mu_{\bar{a}\bar{a}'}(x)$ on peut reprendre l'expression au haut de la page précédente et évaluer les éléments de matrice par rapport au vide, avec $p \neq p'$ (situation sans interaction)

$$\begin{aligned}\left\langle 0 \left| b_3 b_p a_{p'}^\dagger b_1^\dagger \right| 0 \right\rangle &= \left\langle 0 \left| b_3 a_{p'}^\dagger b_p b_1^\dagger \right| 0 \right\rangle = \left\langle 0 \left| a_{p'}^\dagger b_3 b_p b_1^\dagger \right| 0 \right\rangle = 0, \\ \left\langle 0 \left| b_3 a_p^\dagger a_{p'} b_1^\dagger \right| 0 \right\rangle &= \left\langle 0 \left| b_3 a_p^\dagger b_1^\dagger a_{p'} \right| 0 \right\rangle = 0 \\ \left\langle 0 \left| b_3 b_p b_{p'}^\dagger b_1^\dagger \right| 0 \right\rangle &= \left\langle 0 \left| b_3 (b_{p'}^\dagger b_p + \delta_{pp'}) b_1^\dagger \right| 0 \right\rangle = \left\langle 0 \left| b_3 b_{p'}^\dagger b_p b_1^\dagger \right| 0 \right\rangle + \delta_{pp'} \left\langle 0 \left| b_3 b_1^\dagger \right| 0 \right\rangle \\ &= \left\langle 0 \left| (b_{p'}^\dagger b_3 + \delta_{p'3}) b_p b_1^\dagger \right| 0 \right\rangle = \delta_{p'3} \left\langle 0 \left| b_p b_1^\dagger \right| 0 \right\rangle = \delta_{p'3}\delta_{p1} \\ \left\langle 0 \left| b_3 a_p^\dagger b_{p'}^\dagger b_1^\dagger \right| 0 \right\rangle &= \left\langle 0 \left| a_p^\dagger b_3 b_{p'}^\dagger b_1^\dagger \right| 0 \right\rangle = 0\end{aligned}$$

où l'on a noté

$$\delta_{pk} = (2\pi)^3 2E_p \delta^3(p - k).$$

En ne retenant que les valeurs non nulles dans l'intégrale de j^μ on obtient

$$\begin{aligned}\langle p_3 | j^\mu_{\bar{a}\bar{a}'}(x) | p_1 \rangle = -(-e_a) N_1 N_3 \int \frac{d^3p d^3p'}{(2\pi)^6 2E_p 2E_{p'}} (2\pi)^6 2E_p 2E_{p'} \\ \big[- p^\mu \delta^3(p-p_1)\delta^3(p'-p3) e^{-i(p-p')\cdot x} \\ - p'^\mu \delta^3(p-p_1)\delta^3(p'-p_3) e^{-i(p-p')\cdot x} \big]\end{aligned}$$

soit

$$\begin{aligned}\langle p_3 | j^\mu_{\bar{a}\bar{a}'}(x) | p_1 \rangle &= e_a N_1 N_3 \left(-p_1^\mu e^{-i(p_1-p_3)\cdot x} - p_3^\mu e^{-i(p_1-p_3)\cdot x} \right) \\ &= e_a N_1 N_3 (-p_1 - p_3)^\mu e^{-i(p_1-p_3)\cdot x} \\ &= -e_a N_1 N_3 (p_1 + p_3)^\mu e^{-i(p_1-p_3)\cdot x}.\end{aligned}$$

Ainsi l'amplitude de diffusion devient

$$\mathcal{A}_{fi} = -e_b e_a N_2 N_4 N_1 N_3 \int d^4x (p_2+p_4)_\mu \frac{-1}{q^2} e^{-i(p_4-p_2)\cdot x} (p_1+p_3)^\mu e^{-i(p_1-p_3)\cdot x}$$

soit

$$\boxed{\mathcal{A}_{fi} = -e_b e_a N_1 N_2 N_3 N_4 (p_2+p_4)^\mu \frac{-g_{\mu\nu}}{q^2} (p_1+p_3)^\nu (2\pi)^4 \delta^4(p_1+p_2-p_3-p_4).}$$

5.2.4 Problem 5.4 p.170

1. (a) **Dans le référentiel du laboratoire.** On a

$$p_1 = (E_1, \mathbf{p}) \qquad \text{et} \qquad p_2 = (E_2, \mathbf{0}) = (m_2, \mathbf{0})$$

avec

$$E_1^2 = m_1^2 + \mathbf{p}^2.$$

On a alors

$$p_1 \cdot p_2 = E_1 E_2 = m_2 E_1$$

et

$$(p_1 \cdot p_2)^2 = m_2^2 (m_1^2 + \mathbf{p}^2) = m_1^2 m_2^2 + m_2^2 \mathbf{p}^2.$$

Or, on a $\mathbf{p} = \gamma m_1 \mathbf{v}$. On peut alors écrire

$$(p_1 \cdot p_2)^2 - m_1^2 m_2^2 = m_2^2 \gamma^2 m_1^2 \mathbf{v}^2.$$

Mais $\gamma m_1 = E_1$, ainsi

$$(p_1 \cdot p_2)^2 - m_1^2 m_2^2 = E_2^2 E_1^2 \mathbf{v}^2$$

et on a bien

$$\boxed{\left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2 \right]^{\frac{1}{2}} = E_1 E_2 |\mathbf{v}|.}$$

(b) **Dans le référentiel du CM** On a

$$p_1 = (E_1, \mathbf{p}) \qquad \text{et} \qquad p_2 = (E_2, -\mathbf{p}).$$

avec $E_1^2 - \mathbf{p}^2 = m_1^2$ et $E_2^2 - \mathbf{p}^2 = m_2^2$. On a immédiatement

$$p_1 + p_2 = (E_1 + E_2, \mathbf{0})$$

et

$$p_1 \cdot p_2 = E_1 E_2 + \mathbf{p}^2.$$

On a alors

$$(p_1 \cdot p_2)^2 = E_1^2 E_2^2 + \mathbf{p}^2 (2E_1 E_2 + \mathbf{p}^2)$$

et

$$\begin{aligned}(p_1 \cdot p_2)^2 - m_1^2 m_2^2 &= E_1^2 E_2^2 + \mathbf{p}^2(2E_1E_2 + \mathbf{p}^2) - m_1^2 m_2^2 \\ &= E_1^2E_2^2 + 2E_1E_2\mathbf{p}^2 + \mathbf{p}^4 - m_1^2 m_2^3 \\ &= E_1^2E_2^2 + 2E_1E_2\mathbf{p}^2 + (E_2^2 - m_2^2)(E_1^2 - m_1^2) - m_1^2 m_2^3 \\ &= 2E_1^2E_2^2 + 2E_1E_2\mathbf{p}^2 - E_2^2 m_1^2 - E_1^2 m_2^2 \\ &= E_1^2E_2^2 - E_1^2 m_2^2 + E_1^2E_2^2 - E_2^2 m_1^2 + 2E_1E_2\mathbf{p}^2 \\ &= (E_1^2 + E_2^2)\mathbf{p}^2 + 2E_1E_2\mathbf{p}^2 \\ &= (E_1 + E_2)^2\mathbf{p}^2.\end{aligned}$$

Ainsi, on peut écrire

$$(p_1 \cdot p_2)^2 - m_1^2 m_2^2 = (E_1 + E_2)^2\mathbf{p}^2.$$

D'autre part, on remarque que $p_1 \cdot p_2$ est le produit scalaire de deux quadrivecteurs et est donc un scalaire invariant sous les transformations de Lorentz. Les masses sont des scalaires invariants sous les transformations de Lorentz, On a alors, si $\mathbf{v}$ est la vitesse relative des particules projectiles 1 par rapport aux particules cibles 2 (prises dans le référentiel du repos dans le cas du laboratoire, de la partie précédente), que

$$(E_1 + E_2)^2\mathbf{p}^2 = E_1'^2 E_2'^2 |\mathbf{v}|^2.$$

Ainsi, si $f^2 = (p_1 \cdot p_2)^2 - m_1^2 m_2^2$ on a

$$\boxed{f = E_1'E_2'|\mathbf{v}| = (E_1 + E_2)|\mathbf{p}|.}$$

où E_1 et E_2 sont les énergies dans le centre de masse, et E_1' et E_2' sont les énergies dans le référentiel où la particule 2 (ou 1) est au repos.

2. On a

$$p_1^2 = m_1^2 \qquad \text{et} \qquad p_2^2 = m_2^2$$

Avec

$$s = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = m_1^2 + m_2^2 + 2p_1 \cdot p_2$$

on a

$$s - m_1^2 - m_2^2 = 2p_1 \cdot p_2$$

et

$$4(p_1 \cdot p_2)^2 = \left[s - m_1^2 - m_2^2\right] = s^2 + (m_1^2 + m_2^2)^2 - 2s(m_1^2 + m_2^2).$$

Le carré du facteur de flux est alors

$$\begin{aligned}4[(p_1 \cdot p_2)^2 - m_1^2 m_2^2] &= s^2 + (m_1^2 + m_2^2)^2 - 2s(m_1^2 + m_2^2) - 4m_1^2 m_2^2 \\ &= s^2 + (m_1^2 - m_2^2)^2 - 2s(m_1^2 + m_2^2) \\ &= s^2 - 2s(m_1^2 + m_2^2) + (m_1 + m_2)^2(m_1 - m_2)^2 \\ &= [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2].\end{aligned}$$

Ainsi, on a

$$\boxed{4[(p_1 \cdot p_2)^2 - m_1^2 m_2^2] = [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2].}$$

3.

5.2.5 Problem 5.5 p.170

Á calculer l'intégrale (5.145)

$$I = \int e^{iqx}\frac{1}{r}e^{-\mu r}r^2 dr d\Omega.$$

Avec $d\Omega = d(\cos\theta)d\phi$ et en prenant q dans le direction Oz, on obtient, avec 2π pour l'intégrale sur ϕ

$$I = 2\pi \int e^{iqr\cos(\theta)}e^{-\mu r}r\ dr d(\cos\theta)$$

avec $z = \cos\theta \in [-1, 1]$ on a

$$I = 2\pi \int e^{iqrz} e^{-\mu r} r \, dr dz.$$

L'intégrale sur z donne

$$I = 2\pi \int dr \; r e^{-\mu r} \left[\frac{e^{iqrz}}{iqr} \right] = \frac{2\pi}{iq} \int dr \; r e^{-\mu r} \left(e^{-iqr} - e^{iqr} \right).$$

On trouve ainsi

$$I = \frac{2\pi}{iq} \int dr \; e^{-(iq+\mu)r} - e^{(iq-\mu)r} = \frac{2\pi}{iq} \left[\frac{e^{-(iq+\mu)r}}{-(iq+\mu)} - \frac{e^{(iq-\mu)r}}{iq-\mu} \right]_0^\infty$$

soit

$$I = \frac{2\pi}{iq} \left[\frac{1}{iq+\mu} + \frac{1}{iq-\mu} \right]$$

ou encore

$$\boxed{I = \frac{4\pi}{p^2 + \mu^2}.}$$

5.2.6 Problem 5.6 p.170

5.2.7 Problem 5.7 p.170

Chapitre 6

QED pour particules de spins $\frac{1}{2}$

6.1 Problems

6.1.1 Problem 6.1 p.213

1. On normalise un spineur

$$\psi = N \begin{pmatrix} \phi \\ \frac{\sigma \cdot p}{E+m} \phi \end{pmatrix} e^{-ip \cdot x},$$

   dans une boîte de volume V, par

$$\int_V d^3x \ \psi^\dagger \psi = 2E.$$

2. Pour les spineurs u d'énergie positive

$$u(p,s) = (E+m)^{\frac{1}{2}} \begin{pmatrix} \phi^s \\ \frac{\sigma \cdot p}{E+m} \phi^s \end{pmatrix}$$

   en terme des matrices γ on obtient

$$\bar{u}(p,s)(\not{p} - m) = 0$$

3. De même pour les spineurs v d'énergie négative

6.1.2 Problem 6.2 p.214

1. Montrer que pour A et B arbitraire

$$(AB)^\dagger = B^\dagger A^\dagger$$

   où $M^\dagger$ est la matrice conjuguée hermitique de M

$$M^\dagger_{ij} = M^*_{ji}.$$

   On a

$$\begin{aligned} [(AB)^\dagger]_{ij} = (AB)^*_{ji} &= \big(\sum_k A_{jk} B_{ki}\big)^* \\ &= \sum_k A^*_{jk} B^*_{ki} \\ &= \sum_k (A^\dagger)_{kj} (B^\dagger)_{ik} \\ &= \big(A^\dagger B^\dagger\big)_{ij} \end{aligned}$$

2.
3.
4.

6.1.3 Problem 6.3 p.214

6.1.4 Problem 6.4 p.215

6.1.5 Problem 6.5 p.215

6.1.6 Problem 6.6 p.215

6.1.7 Problem 6.7 p.216

6.1.8 Problem 6.8 p.216

6.1.9 Problem 6.9 p.217

6.1.10 Problem 6.10 p.217

6.1.11 Problem 6.11 p.217

6.1.12 Problem 6.12 p.217

6.1.13 Problem 6.13 p.217

Chapitre 7

Diffusion inelastique électron-nucléon

7.1 Problems

7.1.1 Problem 7.1 p.242

7.1.2 Problem 7.2 p.243

7.1.3 Problem 7.3 p.243

7.1.4 Problem 7.4 p.244

7.1.5 Problem 7.5 p.245

Chapitre 8

Non-abelian gauge theory

8.0.1 Problem 8.1 p.280

Selon (8.38) les matrices $\mathbf{T}^{(1)}$ sont données par

$$(T_i^1)_{jk} = -i\epsilon_{ijk},$$

ainsi

$$T_1^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_2^{(1)} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad \text{et} \quad T_3^{(1)} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

On vérifie directement la relation (8.37)

$$[T_i, T_j] = i\epsilon_{ijk} T_k.$$

On a clairement

$$[T_i, T_i] = T_i T_i - T_i T_i = 0.$$

Ensuite

$$T_1 T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$T_2 T_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

ainsi

$$[T_1, T_2] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = iT_3 = i\epsilon_{123} T_3.$$

De même, on a

$$T_2 T_3 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$
$$T_3 T_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

et

$$[T_2, T_3] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} = iT_1 = i\epsilon_{231} T_1$$

et enfin

$$T_3T_1 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$T_1T_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

et

$$[T_3, T_1] = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = iT_2 = i\epsilon_{312}T_2,$$

ce qui achève de vérifier la formule proposée.

8.0.2 Problem 8.2 p.280

Le formule (8.53) donne

$$(1 + i\epsilon \cdot T^{(\frac{1}{2})})\psi^{(\frac{1}{2})}(1 - i\epsilon \cdot T^{(\frac{1}{2})}) = (1 - i\epsilon \cdot \frac{\tau}{2})\psi^{(\frac{1}{2})}$$

et la formule (8.54)

$$[T^{(\frac{1}{2})}, \psi^{(\frac{1}{2})}] = -\frac{\tau}{2}\psi^{(\frac{1}{2})}.$$

Développons le membre de droite de la relation (8.53) au premier ordre en ϵ

$$\psi^{(\frac{1}{2})} + i\epsilon \cdot T^{(\frac{1}{2})}\psi^{(\frac{1}{2})} - i\psi^{(\frac{1}{2})}\epsilon \cdot T^{(\frac{1}{2})} = \psi^{(\frac{1}{2})} - i\epsilon \cdot \frac{\tau}{2}\psi^{(\frac{1}{2})}.$$

On obtient

$$i(\epsilon \cdot T^{(\frac{1}{2})}\psi^{(\frac{1}{2})} - \psi^{(\frac{1}{2})}\epsilon \cdot T^{(\frac{1}{2})}) = -i\epsilon \cdot \frac{\tau}{2}\psi^{(\frac{1}{2})},$$

soit

$$\epsilon_j\left[(T^{(\frac{1}{2})})^j\psi^{(\frac{1}{2})} - \psi^{(\frac{1}{2})}(T^{(\frac{1}{2})})^j\right] = -\epsilon_j\frac{\tau^j}{2}\psi^{(\frac{1}{2})}.$$

Comme les ϵ sont arbitraires, on retrouve bien la formule (8.54).

8.0.3 Problem 8.3 p.280

8.0.4 Problem 8.4 p.280

8.0.5 Problem 8.5 p.280

Chapitre 9

Introduction to QCD

9.0.1 Problem 9.1 p.332

9.0.2 Problem 9.2 p.332

Chapitre 10

Introduction to weak interactions

10.0.1 Problem 10.1 p.356

10.0.2 Problem 10.2 p.356

10.0.3 Problem 10.3 p.357

Chapitre 11

Weak currents

11.0.1 Problem 11.1 p.374

11.0.2 Problem 11.2 p.374

Chapitre 12

Difficulties with weak interaction phenomenology

Chapitre 13

Hidden gauge invariance : $U(1)$

Chapitre 14

The Glashow-Salam-Weinberg theory of electroweak interactions

Chapitre 15

Four last things

Cinquième partie

Peskin and Schroeder

Conventions

D'après le livre de Peskin et Schroeder [PS95]. Les conventions sont les suivantes
Pour la métrique de Minkowski (avec $c = 1$)

$$p^{\mu} = (E, \mathbf{p}), \qquad \partial^{\mu} = (\partial_0, \nabla)$$

et

$$p_{\mu} = (E, -\mathbf{p}), \qquad \partial_{\mu} = (\partial_0, -\nabla).$$

Ainsi

$$p_{\mu} p^{\mu} = E^2 - \mathbf{p}^2 = m^2.$$

Pour les matrices de Dirac

$$\alpha = \begin{pmatrix} \mathbf{0} & \sigma \\ -\sigma & \mathbf{0} \end{pmatrix}$$

et

$$\beta = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

Chapitre 2

The Klein-Gordon Field

2.1 Notes

2.1.1 Élément de volume Lorentz-invariant p.23

Dans un boost de direction selon x^3 on a

$$E' = \gamma(E + \beta p_3) \qquad \text{et} \qquad p'_3 = \gamma(p_3 + \beta E),$$

avec $E'^2 = p_1'^2 + p_2'^2 + p_3'^2 + m^2$ et $E^2 = p_1^2 + p_2^2 + p_3^2 + m^2$ et où $p'_1 = p_1$ et $p'_2 = p_2$. Avec l'identité de la distribution de Dirac

$$\delta(f(x) - f(y)) = \frac{1}{|f'(y)|}\delta(x - y),$$

on a

$$\begin{aligned}
\delta(p - q) &= \delta(p' - q')\frac{dp'_3}{dp_3} \\
&= \delta(p' - q')\gamma(1 + \beta\frac{dE}{dp_3}) \\
&= \delta(p' - q')\frac{\gamma}{E}(E + \beta E\frac{dE}{dp_3}) \\
&= \delta(p' - q')\frac{\gamma}{E}(E + \beta p_3) \\
&= \delta(p' - q')\frac{E'}{E}
\end{aligned}$$

d'où

$$\boxed{E\delta(p - q) = E'\delta(p' - q').}$$

Ainsi, on choisit l'élément de volume Lorentz-invariant

$$\frac{dp}{(2\pi)^2}\frac{1}{2E_p}.$$

Cela conduit à normaliser les états avec

$$\langle p|\, q\rangle = (2\pi)^3 2E_p \delta(p - q)$$

de sorte que

$$\int \frac{dp}{(2\pi)^2}\frac{1}{2E_p}\langle p|\, q\rangle = 1.$$

2.2 Problems

2.2.1 Problem 2.1 p.33

Pour le champ électromagnétique sans source, l'action est donnée par

$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \qquad \text{avec} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

a) Les équations d'Euler-Lagrange sont obtenues en calculant la variation de la densité lagrangienne relativement à A_ρ

$$\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)} \right) - \frac{\delta \mathcal{L}}{\delta A_\mu} = 0$$

avec

$$\begin{aligned}\mathcal{L} = -\frac{1}{4} F_{\sigma\tau} F^{\sigma\tau} &= -\frac{1}{4}(\partial_\sigma A_\tau - \partial_\tau A_\sigma)(\partial^\sigma A^\tau - \partial^\tau A^\sigma) \\ &= -\frac{1}{4}(\partial_\sigma A_\tau - \partial_\tau A_\sigma) g^{\sigma\alpha} g^{\tau\beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha).\end{aligned}$$

Clairement le second terme du membre de gauche de l'équation d'Euler-Lagrange est nul et pour le premier terme on a

$$\begin{aligned}\frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)} = -\frac{1}{4}\big[&\delta_{\sigma\mu}\delta_{\tau\nu} g^{\sigma\alpha} g^{\tau\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\ &- \delta_{\tau\mu}\delta_{\sigma\nu} g^{\sigma\alpha} g^{\tau\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\ &+ (\partial_\sigma A_\tau - \partial_\tau A_\sigma) g^{\sigma\alpha} g^{\tau\beta} \delta_{\alpha\mu}\delta_{\beta\nu} \\ &- (\partial_\sigma A_\tau - \partial_\tau A_\sigma) g^{\sigma\alpha} g^{\tau\beta} \delta_{\alpha\nu}\delta_{\beta\mu}\big],\end{aligned}$$

soit

$$\begin{aligned}\frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)} &= -\frac{1}{4}\big[g^{\mu\alpha} g^{\nu\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\ &\quad - g^{\nu\alpha} g^{\mu\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\ &\quad + (\partial_\sigma A_\tau - \partial_\tau A_\sigma) g^{\sigma\mu} g^{\tau\nu} \\ &\quad - (\partial_\sigma A_\tau - \partial_\tau A_\sigma) g^{\sigma\nu} g^{\tau\mu}\big] \\ &= -\frac{1}{4}\big[(\partial^\mu A^\nu - \partial^\nu A^\mu) - (\partial^\nu A^\mu - \partial^\mu A^\nu) \\ &\quad + (\partial^\mu A^\nu - \partial^\nu A^\mu) - (\partial^\nu A^\mu - \partial^\mu A^\nu)\big] \\ &= -\big[\partial^\mu A^\nu - \partial^\nu A^\mu\big] \\ &= \partial^\nu A^\mu - \partial^\mu A^\nu \\ &= -F^{\mu\nu}.\end{aligned} \tag{2.1}$$

Ainsi, l'équation d'Euler-Lagrange donne

$$\partial_\mu(\partial^\nu A^\mu - \partial^\mu A^\nu) = 0, \qquad \text{ou} \qquad \partial_\mu F^{\mu\nu} = 0,$$

soit

$$\partial^\nu(\partial_\mu A^\mu) - \Box A^\nu = 0,$$

c'est l'équation de Maxwell pour le potentiel

$$\boxed{\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0}, \qquad \text{ou} \qquad \boxed{\partial_\mu F^{\mu\nu} = 0.}$$

En terme de $F^{\mu\nu}$ on a

$$-\partial_\mu F^{\mu\nu} = 0.$$

Ce qui donne, avec c=1,

$$-\partial_t F^{0\nu} + \partial_i F^{i,\nu} = 0.$$

On obtient ainsi

$$\begin{cases} -0 + \partial_i F^{i0} & = 0 \\ -\partial_t F^{0i} + \partial_j F^{ji} & = 0, \end{cases} \quad \text{soit} \quad \begin{cases} \nabla \cdot E & = 0 \\ -\partial_t E + \epsilon^{kji}\partial_j B^k & = 0. \end{cases}$$

où l'on a posé $E^i = F^{0i}$ et $\epsilon^{ijk} B^i = F^{jk}$. On obtient finalement

$$\begin{cases} \nabla \cdot E & = 0 \\ -\partial_t E + \nabla \wedge B & = 0. \end{cases}$$

b)

2.2.2 Problem 2.2 p.33

L'action d'un champ scalaire complexe est donné par

$$S = \int d^4x \, \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi\right).$$

a) La densité lagrangienne de Klein-Gordon est donnée par

$$\boxed{\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi.}$$

Les moments conjugués sont donnés par la variation de $\mathcal{L}$ par rapport aux dérivées temporelles des champs

$$\pi = \frac{\delta \mathcal{L}}{\delta(\partial_0 \phi)} = \partial_0 \phi^*,$$

et

$$\pi^* = \frac{\delta \mathcal{L}}{\delta(\partial_0 \phi^*)} = \partial_0 \phi.$$

Les relations de commutations canoniques sont

$$[\phi(x,t), \pi(y,t)] = i\delta(x-y) \qquad \text{et} \qquad [\phi^*(x,t), \pi^*(y,t)] = i\delta(x-y).$$

La densité hamiltonienne est donnée par

$$\begin{aligned} \mathcal{H} &= \pi \partial_0 \phi + \pi^* \partial_0 \phi^* - \mathcal{L} \\ &= \pi\pi^* + \pi^*\pi - \pi\pi^* + (\nabla\phi^*) \cdot (\nabla\phi) + m^2 \phi^* \phi \end{aligned}$$

soit

$$\boxed{\mathcal{H} = \pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2 \phi^* \phi.}$$

Enfin, les équations du mouvement sont données par les équations d'Euler-Lagrange pour les champs ϕ^* et ϕ

$$\begin{aligned} \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^*)} \right) - \frac{\delta \mathcal{L}}{\delta \phi^*} &= \partial_\mu(\partial^\mu \phi) - (-m^2 \phi) \\ &= \Box\phi + m^2 \phi, \end{aligned}$$

soit l'équation de KG pour ϕ

$$\boxed{\Box\phi + m^2 \phi = 0.}$$

En prenant la variation sur ϕ on trouve

$$\begin{aligned} \partial^\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)} \right) - \frac{\delta \mathcal{L}}{\delta \phi} &= \partial_\mu(\partial^\mu \phi^*) - (-m^2 \phi^*) \\ &= \Box\phi^* + m^2 \phi^*, \end{aligned}$$

soit l'équation de KG pour ϕ^*

$$\boxed{\Box\phi^* + m^2 \phi^* = 0.}$$

b)
c)
d)

2.2.3 Problem 2.3 p.34

Évaluer explicitement la fonction

$$\langle 0 \,| \phi(x)\phi(y) |\, 0 \rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)},$$

pour $x - y$ **de type espace, soit tel que** $(x-y)^2 = -r^2 < 0$**, explicitement en termes des fonctions de Bessel.**

Chapitre 3

The Dirac Field

3.1 Lorentz Invariance in Wave Equations

3.2 The Dirac Equation

3.2.1 p.40

On pose, avec les 4 matrices γ^μ $n \times n$,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} 1_{n\times n}$$

et

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu].$$

Il s'agit de montrer que les matrices $S^{\mu\nu}$ ainsi définies vérifient la relation de commutation des représentations du groupe de Lorentz donnée par la relation (3.17)[PS95, p. 39]

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\big(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\big).$$

On a, (on a simplifié la notation en μ au lieu de γ^μ etc, pour augmenter la lisibilité)

$$\begin{aligned}
[S^{\mu\nu}, S^{\rho\sigma}] &= \frac{i^2}{16}\big[[\gamma^\mu, \gamma^\nu], [\gamma^\rho, \gamma^\sigma]\big] \\
&= \frac{i^2}{16}\big[\mu\nu - \nu\mu, \rho\sigma - \sigma\rho\big] \\
&= \frac{i^2}{16}\big[(\mu\nu\rho\sigma - \mu\nu\sigma\rho - \nu\mu\rho\sigma + \nu\mu\sigma\rho) - (\rho\sigma\mu\nu - \rho\sigma\nu\mu - \sigma\rho\mu\nu + \sigma\rho\nu\mu)\big] \\
&= \frac{i^2}{16}\big[\mu\nu\rho\sigma - \mu\nu\sigma\rho - \nu\mu\rho\sigma + \nu\mu\sigma\rho - \rho\sigma\mu\nu + \rho\sigma\nu\mu + \sigma\rho\mu\nu - \sigma\rho\nu\mu \\
&\quad + \mu\rho\nu\sigma - \mu\sigma\nu\rho - \nu\rho\mu\sigma + \nu\sigma\mu\rho - \rho\mu\sigma\nu + \rho\nu\sigma\mu + \sigma\mu\rho\nu - \sigma\nu\rho\mu \\
&\quad - \mu\rho\nu\sigma + \mu\sigma\nu\rho + \nu\rho\mu\sigma - \nu\sigma\mu\rho + \rho\mu\sigma\nu - \rho\nu\sigma\mu - \sigma\mu\rho\nu + \sigma\nu\rho\mu\big]
\end{aligned}$$

Chacun des termes des deux premières lignes de la dernière égalité peuvent être appariés et font apparaître la relation d'anticommutation des matrices γ^μ, on obtient alors

$$\begin{aligned}
&[S^{\mu\nu}, S^{\rho\sigma}] \\
&\quad= \frac{i^2}{16}\big[2g^{\nu\rho}\mu\sigma - 2g^{\nu\sigma}\mu\rho - 2g^{\mu\rho}\nu\sigma + 2g^{\mu\sigma}\nu\rho - 2g^{\mu\sigma}\rho\nu + 2g^{\nu\sigma}\rho\mu \\
&\qquad + 2g^{\mu\rho}\sigma\nu - 2g^{\nu\rho}\sigma\mu\mu\rho\nu\sigma + \mu\sigma\nu\rho + \nu\rho\mu\sigma - \nu\sigma\mu\rho + \rho\mu\sigma\nu \\
&\qquad - \rho\nu\sigma\mu - \sigma\mu\rho\nu + \sigma\nu\rho\mu\big]
\end{aligned}$$

En réitérant l'opération, on trouve

$$\begin{aligned}
&[S^{\mu\nu}, S^{\rho\sigma}] \\
&= \frac{i^2}{16}\big[2g^{\nu\rho}[\mu,\sigma] - 2g^{\nu\sigma}[\mu,\rho] - 2g^{\mu\rho}[\nu,\sigma] + 2g^{\mu\sigma}[\nu,\rho] \\
&\qquad - \mu\rho\nu\sigma + \mu\sigma\nu\rho + \nu\rho\mu\sigma - \nu\sigma\mu\rho + \rho\mu\sigma\nu - \rho\nu\sigma\mu - \sigma\mu\rho\nu + \sigma\nu\rho\mu \\
&\qquad - \rho\mu\nu\sigma + \sigma\mu\nu\rho + \rho\nu\mu\sigma - \sigma\nu\mu\rho + \rho\mu\nu\sigma - \sigma\mu\nu\rho - \rho\nu\mu\sigma + \sigma\nu\mu\rho\big] \\
&= \frac{i^2}{16}\big[4g^{\nu\rho}[\mu,\sigma] - 4g^{\nu\sigma}[\mu,\rho] - 4g^{\mu\rho}[\nu,\sigma] + 4g^{\mu\sigma}[\nu,\rho]\big] \\
&= \frac{i^2}{16}\frac{16}{i}\big(g^{\nu\rho}S^{\mu\sigma} - g^{\nu\sigma}S^{\mu\rho} - g^{\mu\rho}S^{\nu\sigma} + g^{\mu\sigma}S^{\nu\rho}\big) \\
&= i\big(g^{\nu\rho}S^{\mu\sigma} - g^{\nu\sigma}S^{\mu\rho} - g^{\mu\rho}S^{\nu\sigma} + g^{\mu\sigma}S^{\nu\rho}\big).
\end{aligned}$$

Ainsi, les matrices $S^{\mu\nu}$ forment bien une représentation de l'algèbre de Lorentz.

3.7 Problems

3.7.3 Problem 3.3 p.72

Soit

$$k_0^2 = 0, \qquad k_1^2 = -1, \qquad \text{avec} \quad k_0 \cdot k_1 = 0.$$

Soit encore

$$p^2 = 0.$$

Soit $u_{L0}(k_0)$ le spineur de Dirac *gauche* de masse nulle et d'impulsion k_0. On définit

$$u_{R0} = \not{k}_1 u_{L0}, \qquad u_L(p) = \frac{1}{\sqrt{2p\cdot k_0}}\ \not{p}u_{R0} \qquad \text{et} \quad u_R(p) = \frac{1}{\sqrt{2p\cdot k_0}}\ \not{p}u_{L0}.$$

1. Montrer que $\not{k}_0 u_{R0} = \not{p}u_L(p) = \not{p}u_R(p) = 0$. Clairement, vu que $u_{L0}(k_0)$ est un spineur de Dirac d'impulsion k_0 de masse nulle, il doit satisfaire l'équation de Dirac de masse nulle

$$\not{k}_0 u_{L0}(k_0) = 0.$$

On peut alors calculer

$$\begin{aligned}
\not{k}_0 u_{R0} &= \not{k}_0\ \not{k}_1 u_{L0} \\
&= k_0^\mu k_1^\nu \gamma_\mu\gamma_\nu u_{L0} \\
&= k_0^\mu k_1^\nu (2g_{\mu\nu} - \gamma_\nu\gamma_\mu) u_{L0} \\
&= 2k_0\cdot k_1 u_{L0} - \not{k}_1\ \not{k}_0 u_{L0} \\
&= 0 u_{L0} - \not{k}_1 0 = 0
\end{aligned}$$

2. D'autre part, on a

$$\begin{aligned}
\not{p}u_L(p) &= \not{p}\frac{1}{\sqrt{2p\cdot k_0}}\ \not{p}u_{R0} \\
&= \frac{\not{p}\ \not{p}}{\sqrt{2p\cdot k_0}} u_{R0} \\
&= \frac{p^\mu p^\nu}{\sqrt{2p\cdot k_0}} \gamma_\mu \gamma \nu u_{R0} \\
&= \frac{p^\mu p^\nu}{\sqrt{2p\cdot k_0}} (2g_{\mu\nu} - \gamma_\nu \gamma \mu) u_{R0} \\
&= -\frac{p^\mu p^\nu}{\sqrt{2p\cdot k_0}} \gamma_\nu \gamma \mu u_{R0} \qquad (\text{car} \quad p^2 = 0) \\
&= -\frac{\not{p}\ \not{p}}{\sqrt{2p\cdot k_0}} u_{R0}.
\end{aligned}$$

On en déduit alors

$$2\frac{\not p\,\not p}{\sqrt{2p\cdot k_0}}u_{R0}=0$$

et ainsi

$$\not p u_L(p)=0.$$

On procède exactement de la même façon pour montrer que

$$\not p u_R(p)=0.$$

3. On pose $k_0=(E,0,0,-E)$ et $k_1=(0,1,0,0)$. Le spineur gauche sans masse $u_{L0}(k_0)$ s'écrit alors

$$u_{L0}(k_0)=\begin{pmatrix}0\\0\\0\\\sqrt{2E}\end{pmatrix}.$$

On obtient

$$\begin{aligned}u_{R0}&=\not k_1 u_{L0}=-\gamma_2 u_{L0}\\&=-\begin{pmatrix}0&\sigma_2\\-\sigma_2&0\end{pmatrix}u_{L0}\\&=-\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}u_{L0}\\&=\begin{pmatrix}i\sqrt{2E}\\0\\0\\0\end{pmatrix}.\end{aligned}$$

Pour construire $u_L(p)$ et $u_R(p)$ on utilise

$$\not p=p_\mu\gamma^\mu=p^\mu\begin{pmatrix}0&\sigma_\mu\\\bar\sigma_\mu&0\end{pmatrix}=\begin{pmatrix}0&0&p^0-p^3&-p^1+ip^2\\0&0&-p^1-ip^2&p^0+p^3\\p^0+p^3&p^1-ip^2&0&0\\p^1+ip^2&p^0-p^3&0&0\end{pmatrix}$$

et on obtient

$$u_L(p)=\frac{1}{\sqrt{2p\cdot k_0}}u_{R0}=\frac{i}{\sqrt{2E(p^0+p^3)}}\sqrt{2E}\begin{pmatrix}0\\0\\p^0+p^3\\p^1+ip^2\end{pmatrix}$$

soit

$$\boxed{u_L(p)=\frac{i}{\sqrt{p^0+p^3}}\begin{pmatrix}0\\0\\p^0+p^3\\p^1+ip^2\end{pmatrix}}$$

et

$$u_R(p)=\frac{1}{\sqrt{2p\cdot k_0}}u_{L0}=\frac{1}{\sqrt{2E(p^0+p^3)}}\sqrt{2E}\begin{pmatrix}-p^1+ip^2\\p^0+p^3\\0\\0\end{pmatrix}$$

soit

$$\boxed{u_R(p)=\frac{1}{\sqrt{p^0+p^3}}\begin{pmatrix}-p^1+ip^2\\p^0+p^3\\0\\0\end{pmatrix}}.$$

4. Calculer $s(p,k)=\bar u_R(p)u_L(k)$, $s(p,k)^*$, $|s(p,k)|^2$ et $t(p,k)=\bar u_L(p)u_R(k)$.

Chapitre 4

Interacting Fields and Feynman Diagrams

Sixième partie

S. Weinberg I. Foundations

Conventions

D'après le livre de S. Weinberg [Wei95]. mais la convention utilisée pour les solutions est

$$\eta = \text{diag}(1, -1, -1, -1)$$

dans l'ordre des indices $0, 1, 2, 3$ et la répétition des indices définit le produit scalaire

$$p \cdot x = p^\mu x_\mu = p^0 x^0 - p^1 x^1 - p^2 x^2 - p^3 x^3.$$

Chapitre 2

Relativistic Quantum Mechanics

Exercise 2.1

Suppose that observer $\mathcal{O}$ sees a W-boson (spin one and mass $m \neq 0$) with momentum $\mathbf{p}$ in the y-direction and spin z-component σ. A second observer $\mathcal{O}'$ moves relative to the first with velocity $\mathbf{v}$ in the z-direction. How does $\mathcal{O}'$ describe the W state ?

Take $\bar{k} = k^\mu = (M, 0, 0, 0)$, $\bar{p} = p^\mu = (E, 0, p, 0)$ with $E = \sqrt{m^2c^4 + p^2c^2}$ and

$$\beta = \frac{v}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}},$$

and (with $c = 1$)

$$\gamma_p = \frac{E}{M}, \qquad \text{with} \quad E^2 = M^2 + p^2.$$

Then

$$\gamma^2 - 1 = \beta\gamma^2.$$

The boost that bring the W-state from its rest frame where its momentum is k to the frame of observer $\mathcal{O}$ where its momentum is p^μ is then given by (2.5.24, page 68, vol.I)

$$\begin{aligned} L_0{}^0(p) &= \gamma_p \\ L_0{}^i(p) &= \hat{p}^i\sqrt{\gamma_p^2 - 1} \\ L_i{}^j(p) &= \delta_{ij} + (\gamma_p - 1)\hat{p}^i\hat{p}^j \end{aligned}$$

that is

$$L(p) = \begin{pmatrix} \frac{E}{M} & 0 & \frac{p}{M} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{p}{M} & 0 & \frac{E}{M} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma_p & 0 & \beta_p\gamma_p & 0 \\ 0 & 1 & 0 & 0 \\ \beta_p\gamma_p & 0 & \gamma_p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so that $L(p)\bar{k} = \bar{p}$. The boost that brings the frame of $\mathcal{O}$ to the frame of $\mathcal{O}'$ is

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

We can then check that

$$\det(L(p)) = \det(\Lambda) = 1.$$

The state described by $\mathcal{O}$ is labeled by p and σ and is obtained by applying the unitary operator $U(L(p))$ to the state $\Psi_{\bar{k},\sigma'}$ (see eq. (2.5.5)–(2.5.8) on page 64, vol.I)

$$\Psi_{\bar{p},\sigma} = N(p)U(L(p))\Psi_{\bar{k},\sigma} = N(p)\sum_{\sigma'} D_{\sigma'\sigma}(L(p))\Psi_{L(p)\bar{k},\sigma'}$$

with $N(p) = \sqrt{\frac{k_0}{p_0}}$.

The combination of the two boosts gives

$$\Lambda L(p) = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \gamma_p & 0 & \beta_p\gamma_p & 0 \\ 0 & 1 & 0 & 0 \\ \beta_p\gamma_p & 0 & \gamma_p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which gives

$$\Lambda L(p) = \begin{pmatrix} \gamma\gamma_p & 0 & \frac{p}{M} & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ \frac{p}{M} & 0 & \gamma_p & 0 \\ -\beta\gamma\gamma_p & 0 & -\beta\gamma\frac{p}{M} & \gamma \end{pmatrix} = \begin{pmatrix} \gamma\gamma_p & 0 & \beta_p\gamma_p\gamma & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ \beta_p\gamma_p & 0 & \gamma_p & 0 \\ -\beta\gamma\gamma_p & 0 & -\beta\beta_p\gamma_p\gamma & \gamma \end{pmatrix}$$

The boost Λ applied on the 4-vector $\bar{p}$ gives

$$\Lambda\bar{p} = \begin{pmatrix} \gamma E \\ p \\ 0 \\ -\beta\gamma E \end{pmatrix}$$

and the boost that bring k to $\Lambda\bar{p}$ is then

$$L(\Lambda\bar{p}) = \begin{pmatrix} \gamma\gamma_p & 0 & \frac{p}{M} & -\beta\gamma\gamma_p \\ 0 & 1 & 0 & 0 \\ \frac{p}{M} & 0 & 1+(\gamma\gamma_p-1)\frac{p^2}{p^2+\beta^2\gamma^2E^2} & (\gamma\gamma_p-1)\frac{-\beta\gamma pE}{p^2+\beta^2\gamma^2E^2} \\ -\beta\gamma\gamma_p & 0 & (\gamma\gamma_p-1)\frac{-\beta\gamma pE}{p^2+\beta^2\gamma^2E^2} & 1+(\gamma\gamma_p-1)\frac{\beta^2\gamma^2E^2}{p^2+\beta^2\gamma^2E^2} \end{pmatrix}$$

After some rearrangements we can write it as

$$L(\Lambda\bar{p}) = \begin{pmatrix} \gamma\gamma_p & 0 & \beta_p\gamma_p & -\beta\gamma\gamma_p \\ 0 & 1 & 0 & 0 \\ \beta_p\gamma_p & 0 & \gamma_p\frac{\gamma+\gamma_p}{\gamma\gamma_p+1} & -\frac{\gamma\beta\beta_p\gamma_p^2}{\gamma\gamma_p+1} \\ -\beta\gamma\gamma_p & 0 & -\frac{\gamma\beta\beta_p\gamma_p^2}{\gamma\gamma_p+1} & 1+\frac{(\gamma^2-1)\gamma_p^2}{\gamma\gamma_p+1} \end{pmatrix}$$

The inverse of $L(\Lambda\bar{p})$ is then

$$L(\Lambda\bar{p})^{-1} = \begin{pmatrix} \gamma\gamma_p & 0 & -\beta_p\gamma_p & \beta\gamma\gamma_p \\ 0 & 1 & 0 & 0 \\ -\beta_p\gamma_p & 0 & \gamma_p\frac{\gamma+\gamma_p}{\gamma\gamma_p+1} & -\frac{\gamma\beta\beta_p\gamma_p^2}{\gamma\gamma_p+1} \\ \beta\gamma\gamma_p & 0 & -\frac{\gamma\beta\beta_p\gamma_p^2}{\gamma\gamma_p+1} & 1+\frac{(\gamma^2-1)\gamma_p^2}{\gamma\gamma_p+1} \end{pmatrix}$$

We can now compute the Wigner rotation

$$W = L(\Lambda\bar{p})^{-1}\Lambda L(\bar{p}) = \begin{pmatrix} \gamma\gamma_p & 0 & -\beta_p\gamma_p & \beta\gamma\gamma_p \\ 0 & 1 & 0 & 0 \\ -\beta_p\gamma_p & 0 & \gamma_p\frac{\gamma+\gamma_p}{\gamma\gamma_p+1} & -\frac{\gamma\beta\beta_p\gamma_p^2}{\gamma\gamma_p+1} \\ \beta\gamma\gamma_p & 0 & -\frac{\gamma\beta\beta_p\gamma_p^2}{\gamma\gamma_p+1} & 1+\frac{(\gamma^2-1)\gamma_p^2}{\gamma\gamma_p+1} \end{pmatrix} \begin{pmatrix} \gamma\gamma_p & 0 & \beta_p\gamma_p\gamma & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ \beta_p\gamma_p & 0 & \gamma_p & 0 \\ -\beta\gamma\gamma_p & 0 & -\beta\beta_p\gamma_p\gamma & \gamma \end{pmatrix}$$

which gives after some simplifications

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\gamma+\gamma_p}{\gamma\gamma_p+1} & \frac{\gamma\gamma_p\beta\beta_p}{\gamma\gamma_p+1} \\ 0 & 0 & -\frac{\gamma\gamma_p\beta\beta_p}{\gamma\gamma_p+1} & \frac{\gamma+\gamma_p}{\gamma\gamma_p+1} \end{pmatrix}$$

this is clearly a rotation by an angle θ around the x-axis such that

$$\tan(\theta) = \frac{\gamma\gamma_p\beta\beta_p}{\gamma+\gamma_p} = \frac{\sqrt{(\gamma^2-1)(\gamma_p^2-1)}}{\gamma+\gamma_p}$$

The operator D is then given by the generator of the rotation around the x-axis J_x for spin-1 by

$$D(W) = e^{i\theta J_x}$$

with

$$J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and we get

$$D(W) = \begin{pmatrix} \frac{1}{2}(\cos(\theta)+1) & \frac{i}{\sqrt{2}}\sin(\theta) & \frac{1}{2}(\cos(\theta)-1) \\ \frac{i}{\sqrt{2}}\sin(\theta) & \cos(\theta) & \frac{i}{\sqrt{2}}\sin(\theta) \\ \frac{1}{2}(\cos(\theta)-1) & \frac{i}{\sqrt{2}}\sin(\theta) & \frac{1}{2}(\cos(\theta)+1) \end{pmatrix}$$

We can now write the state of the W-boson in the frame $\mathcal{O}'$, with the normalization factor $\sqrt{\frac{(\Lambda\bar{p})^0}{\bar{p}^0}} = \sqrt{\gamma}$, using eq (2.5.23, page 68, vol.I)

$$\Psi'_{\Lambda\bar{p},\sigma'} = U(\Lambda)\Psi_{p,\sigma} = \sqrt{\gamma}\sum_\sigma D(W)_{\sigma\sigma'}\Psi_{\Lambda\bar{p},\sigma}$$

For vanishing speeds (non-relativistic limit) we have (restoring the c's)

$$\tan(\theta) \simeq \frac{1}{2}\frac{vv_p}{c^2} = \frac{1}{2}\frac{vp}{Mc^2} \simeq \theta$$

for small θ. In that case

$$D(W) = \begin{pmatrix} 1-\frac{1}{4}\theta^2 & \frac{i\theta}{\sqrt{2}} & -\frac{1}{4}\theta^2 \\ \frac{i\theta}{\sqrt{2}} & 1-\frac{1}{2}\theta^2 & \frac{i\theta}{\sqrt{2}} \\ -\frac{1}{4}\theta^2 & \frac{i\theta}{\sqrt{2}} & 1-\frac{1}{4}\theta^2 \end{pmatrix}$$

and, for example, the 1-spin state $\sigma = (0,0,1)^T$ is transformed, to second order, to a state

$$D(W)\bar{\sigma} = \begin{pmatrix} -\dfrac{\theta^2}{4} \\ \dfrac{i\theta}{\sqrt{2}} \\ 1-\dfrac{1}{4}\theta^2 \end{pmatrix}$$

Exercise 2.2

Suppose that observer $\mathcal{O}$ sees a photon with momentum $\mathbf{p}$ in the y-direction and polarization vector in the z-direction. A second observer $\mathcal{O}'$ moves relative to the first with velocity $\mathbf{v}$ in the z-direction. How does $\mathcal{O}'$ describe the same photon ?

The photon being massless, the helicity of the photon is invariant. So if the state of the photon for $\mathcal{O}$ is

$$\Psi_{\bar{p},\sigma}$$

the state of that same photon in the frame $\mathcal{O}'$ will be given by the same helicity state but a change in phase (2.5.42, page 72 vol.I)

$$\Psi'_{\Lambda\bar{p},\sigma} = U(W)\Psi_{\bar{p}.\sigma} = \sqrt{\gamma}e^{i\sigma\theta(\Lambda,\bar{p})}\Psi_{\Lambda\bar{p},\sigma}$$

with θ defined by (2.5.43, page 72, vol.I)

$$W(\Lambda,\bar{p}) = L^{-1}(\Lambda\bar{p})\Lambda L(\bar{p}) = S(\alpha(\Lambda,\bar{p}),\beta(\Lambda,\bar{p}))R(\theta)$$

With $\bar{k} = (k,0,0,k)^T$ and $\bar{p} = (p,0,p,0)^T$ we define the matrices

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$L(\bar{p})$ given by (2.5.44, page 73) as $L(\bar{p}) = RB$ with

$$B = \begin{pmatrix} \frac{p^2+k^2}{2pk} & 0 & 0 & \frac{p^2-k^2}{2pk} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{p^2-k^2}{2pk} & 0 & 0 & \frac{p^2+k^2}{2pk} \end{pmatrix}$$

and

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Then

$$\Lambda RB = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \frac{p^2+k^2}{2pk} & 0 & 0 & \frac{p^2-k^2}{2pk} \\ 0 & 1 & 0 & 0 \\ \frac{p^2-k^2}{2pk} & 0 & 0 & \frac{p^2+k^2}{2pk} \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} \gamma a_+ & 0 & \beta\gamma & \gamma a_- \\ 0 & 1 & 0 & 0 \\ a_- & 0 & 0 & a_+ \\ -\beta\gamma a_+ & 0 & -\gamma & -\beta\gamma a_- \end{pmatrix}$$

where $a_\pm = \dfrac{p^2 \pm k^2}{2pk}$.
And with $\Lambda\bar{p} = (\gamma p, 0, p, -\beta\gamma p)^T$, we define $L(\Lambda\bar{p}) = R'B'$ with

$$B'(\Lambda\bar{p}) = \begin{pmatrix} \frac{\gamma^2p^2+k^2}{2\gamma pk} & 0 & 0 & \frac{\gamma^2p^2-k^2}{2\gamma pk} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\gamma^2p^2-k^2}{2\gamma pk} & 0 & 0 & \frac{\gamma^2p^2+k^2}{2\gamma pk} \end{pmatrix} \Leftrightarrow B'(\Lambda\bar{p})^{-1} = \begin{pmatrix} \frac{\gamma^2p^2+k^2}{2\gamma pk} & 0 & 0 & -\frac{\gamma^2p^2-k^2}{2\gamma pk} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma^2p^2-k^2}{2\gamma pk} & 0 & 0 & \frac{\gamma^2p^2+k^2}{2\gamma pk} \end{pmatrix}$$

the boost that brings $(k,0,0,k)$ to $(\gamma p,0,0,\gamma p)$ that we may write as

$$B'(\Lambda\bar{p})^{-1} = \begin{pmatrix} b_+ & 0 & 0 & -b_- \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -b_- & 0 & 0 & b_+ \end{pmatrix}$$

where $b_\pm = \dfrac{\gamma^2 p^2 \pm k^2}{2\gamma p k}$, and

$$R'(\Lambda\bar{p}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\beta & \frac{1}{\gamma} \\ 0 & 0 & -\frac{1}{\gamma} & -\beta \end{pmatrix} \Leftrightarrow R'(\Lambda\bar{p})^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\beta & -\frac{1}{\gamma} \\ 0 & 0 & \frac{1}{\gamma} & -\beta \end{pmatrix}$$

the rotation that bring $(\gamma p, 0, 0, \gamma p)$ to $\Lambda\bar{p}$. So, according to [Wei95, (2.5.43),p.73]

$$\begin{aligned} W &= L^{-1}(\Lambda\bar{p})\Lambda L(\bar{p}) \\ &= B'^{-1}R'^{-1}\Lambda R B \\ &= \begin{pmatrix} b_+ & 0 & -\frac{1}{\gamma}b_- & \beta b_- \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\beta & -\frac{1}{\gamma} \\ -b_- & 0 & \frac{1}{\gamma}b_+ & -\beta b_+ \end{pmatrix} \begin{pmatrix} \gamma a_+ & 0 & \beta\gamma & \gamma a_- \\ 0 & 1 & 0 & 0 \\ a_- & 0 & 0 & a_+ \\ -\beta\gamma a_+ & 0 & -\gamma & -\beta\gamma a_- \end{pmatrix} \end{aligned}$$

we obtain thus

$$\begin{aligned} W &= \begin{pmatrix} \gamma a_+ b_+ - \frac{1}{\gamma}a_- b_- - \beta^2\gamma a_+ b_- & 0 & \beta\gamma(b_+ - b_-) & \gamma a_- b_+ - \frac{1}{\gamma}a_+ b_- \beta^2\gamma a_- b_- \\ 0 & 1 & 0 & 0 \\ \beta(a_+ - a_-) & 0 & 1 & -\beta(a_+ - a_-) \\ -\gamma a_+ b_- + \frac{1}{\gamma}a_- b_+ + \beta^2\gamma a_+ b_+ & 0 & \beta\gamma(b_+ - b_-) & -\gamma a_- b_- + \frac{1}{\gamma}a_+ b_+ + \beta^2\gamma a_- b_+ \end{pmatrix} \\ &= S(a,b)R''(\theta) \end{aligned}$$

Reducing the expression for W we get

$$W = \begin{pmatrix} 1 + \frac{\beta^2 k^2}{2p^2} & 0 & \beta\frac{k}{p} & -\frac{\beta^2 k^2}{2p^2} \\ 0 & 1 & 0 & 0 \\ \beta\frac{k}{p} & 0 & 1 & -\beta\frac{k}{p} \\ \frac{\beta^2 k^2}{2p^2} & 0 & \beta\frac{k}{p} & 1 - \frac{\beta^2 k^2}{2p^2} \end{pmatrix}$$

With a Lorentz transform S , where $c = \dfrac{a^2 + b^2}{2}$ and a rotation R'' around the x-axis, given by

$$S(a,b) = \begin{pmatrix} 1+c & a & b & -c \\ a & 1 & 0 & -a \\ b & 0 & 1 & -b \\ c & a & b & 1-c \end{pmatrix} \qquad R'' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t_1 & t_2 \\ 0 & 0 & -t_2 & t_1 \end{pmatrix}$$

The matrix W should be written as

$$W = \begin{pmatrix} 1+c & a & bt_1 + ct_2 & bt_2 - ct_1 \\ a & 1 & at_2 & -at_1 \\ b & 0 & t_1 + bt_2 & t_2 - bt_1 \\ c & a & bt_1 - (1-c)t_2 & bt_2 + (1-c)t_1 \end{pmatrix}$$

By identification with the previous expression we deduce $a = 0$ and $b = \beta\frac{k}{p}$

$$S(0,b) = \begin{pmatrix} 1 + \frac{\beta^2 k^2}{2p^2} & 0 & \beta\frac{k}{p} & \frac{\beta^2 k^2}{2p^2} \\ 0 & 1 & 0 & 0 \\ \beta\frac{k}{p} & 0 & 1 & -\beta\frac{k}{p} \\ \frac{\beta^2 k^2}{2p^2} & 0 & \beta\frac{k}{p} & 1 - \frac{\beta^2 k^2}{2p^2} \end{pmatrix}$$

So that the rotation matrix is the identity, and we can take $\theta = 0$, with $t_1 = 1$ and $t_2 = 0$.

In the frame $\mathcal{O}'$ the photon is describe by

$$\Psi'_{\Lambda\bar{p},\sigma} = \sqrt{\gamma}\Psi_{\Lambda\bar{p},\sigma}$$

Exercise 2.3

Derive the commutation relations for the generators of the Galilean group directly from the group multiplication law. Include the most general set of central charges that cannot be eliminated by redefinition of the group generators.

Exercise 2.4

Show that the operators $P_\mu P^\mu$ and $W_\mu W^\mu$ commute with all Lorentz transformation operators $U(\Lambda, a)$ where $W_\mu = \epsilon_{\mu\nu\rho\lambda} J^{\nu\rho} P^\lambda$.

Exercise 2.5

Consider physics in two space and one time dimensions, assuming invariance under a 'Lorentz' group $SO(1,2)$. How would you describe the spin states of a single massive particle ? How do they behave under Lorentz transformations ? What about the inversions P ans T ?

In two space dimensions, P is an element of $SO(1,2)$ (it is simply a rotation by $180°$), but T is not. The matrices are respectively

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \mathcal{T} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If a state is described by $\Psi_{\bar{p},\sigma}$ the state $\Psi'_{\mathbf{p}',\sigma'} = \mathsf{P}\Psi_{\mathbf{p},\sigma} = \eta\Psi_{-\mathbf{p},\sigma}$

Exercise 2.6

As in problem 5, consider physics in two space and one time dimensions, assuming invariance under a 'Lorentz' group $O(1,2)$. How would you describe the spin states of a single massless particle ? How do they behave under Lorentz transformations ? What about the inversions P ans T ?

Chapitre 3

Scattering Theory

Exercise 3.1

Consider a theory with a separable interaction ; that is

$$(\Phi_\beta, V\Phi_\alpha) = g u_\beta u_\alpha^*,$$

where g is a real coupling constante, and u_α is a set of complex quantities with

$$\sum_\alpha |u_\alpha|^2 = 1.$$

Use the Lippmann-Schwinger equation (3.1.16) to find explicit solutions for the 'in' and 'out' states and the S-matrix.

The Lippmann-Schwinger equations state that

$$\Psi_\alpha^\pm = \Phi_\alpha + \frac{V\Psi_\alpha^\pm}{E_\alpha - H_0 \pm i\epsilon}.$$

By multiplying by V and taking the Φ_β component we have (by letting H_0 act on the right)

$$(\Phi_\beta, V\Psi_\alpha^\pm) = (\Phi_\beta, V\Phi_\alpha) + \frac{(\Phi_\beta, V^2\Psi_\alpha^\pm)}{E_\alpha - E_\beta \pm i\epsilon}$$

Using the closure relation we have

$$(\Phi_\beta, V\Psi_\alpha^\pm) = (\Phi_\beta, V\Phi_\alpha) + \int d\gamma \frac{(\Phi_\beta, V\Phi_\gamma)(\Phi_\gamma, V\Psi_\alpha^\pm)}{E_\alpha - E_\beta \pm i\epsilon}$$

and with the definition of separable interaction and the definition of $T_{\beta\alpha}^\pm = (\Phi_\beta, V\Psi_\alpha^\pm)$

$$\begin{aligned} T_{\beta\alpha}^\pm &= g u_\beta u_\alpha^* + \frac{g u_\beta}{E_\alpha - E_\beta \pm i\epsilon} \int d\gamma \; u_\gamma^* T_{\gamma\alpha}^\pm \\ &= g u_\beta \left(u_\alpha^* + \frac{1}{E_\alpha - E_\beta \pm i\epsilon} \int d\gamma \; u_\gamma^* T_{\gamma\alpha}^\pm \right) \end{aligned}$$

if we reinsert that expression for $T_{\gamma\alpha}^\pm$ we have

$$\begin{aligned} T_{\beta\alpha}^\pm &= g u_\beta \left(u_\alpha^* + \frac{1}{E_\alpha - E_\beta \pm i\epsilon} \int d\gamma \; u_\gamma^* g u_\gamma \left(u_\alpha^* + \frac{1}{E_\alpha - E_\gamma \pm i\epsilon} \int d\rho \; u_\rho^* T_{\rho\alpha}^\pm \right) \right) \\ &= g u_\beta \left(u_\alpha^* + \frac{g u_\alpha^*}{E_\alpha - E_\beta \pm i\epsilon} + \frac{g}{E_\alpha - E_\beta \pm i\epsilon} \int d\gamma \; \frac{|u_\gamma|^2}{E_\alpha - E_\gamma \pm i\epsilon} \int d\rho \; u_\rho^* T_{\rho\alpha}^\pm \right) \end{aligned}$$

where it remains to calculate

$$I_\alpha = \int d\gamma\ \frac{|u_\gamma|^2}{E_\alpha - E_\gamma \pm i\epsilon} \int d\rho\ u_\rho^* T_{\rho\alpha}^\pm$$

Define

$$U_{\beta\alpha} = \frac{T_{\beta\alpha}^\pm}{u_\alpha^*}$$

then

$$U_{\beta\alpha} = g u_\beta \left(1 + \frac{1}{E_\alpha - E_\beta \pm i\epsilon} \int d\gamma\ u_\gamma^* U_{\gamma\alpha}\right)$$

so

$$\frac{I_\alpha}{u_\alpha^*} = \int d\gamma\ \frac{|u_\gamma|^2}{E_\alpha - E_\gamma \pm i\epsilon} \int d\rho\ u_\rho^* U_{\rho\alpha}$$

and iterating

$$\begin{aligned}\frac{I_\alpha}{u_\alpha^*} &= g \int d\gamma\ \frac{|u_\gamma|^2}{E_\alpha - E_\gamma \pm i\epsilon} \int d\rho\ u_\rho^* u_\rho \left(1 + \frac{1}{E_\alpha - E_\rho \pm i\epsilon} \int d\sigma\ u_\sigma^* U_{\sigma\alpha}\right) \\ &= g \int d\gamma\ \frac{|u_\gamma|^2}{E_\alpha - E_\gamma \pm i\epsilon} \left(1 + \int d\rho\ \frac{|u_\rho|^2}{E_\alpha - E_\rho \pm i\epsilon} \int d\sigma\ u_\sigma^* U_{\sigma\alpha}\right)\end{aligned}$$

Define

$$\boxed{k_\alpha^\pm = g \int d\gamma \frac{|u_\gamma|^2}{E_\alpha - E_\gamma \pm i\epsilon}}$$

, from which we see that $(k_\alpha^+)^* = k_\alpha^-$. We have then

$$\frac{I_\alpha}{u_\alpha^*} = k_\alpha^\pm + k_\alpha^\pm \frac{I_\alpha}{u_\alpha^*}$$

from which we have

$$I_\alpha = u_\alpha^* \frac{k_\alpha^\pm}{1 - k_\alpha^\pm}$$

So, finally, we can write

$$T_{\beta\alpha}^\pm = g u_\beta u_\alpha^* \left(1 + \frac{g}{E_\alpha - E_\beta \pm i\epsilon}\left(1 + \frac{k_\alpha^\pm}{1 - k_\alpha^\pm}\right)\right)$$

that is

$$\boxed{T_{\beta\alpha}^\pm = g u_\beta u_\alpha^* \left(1 + \frac{g}{E_\alpha - E_\beta \pm i\epsilon} \frac{1}{1 - k_\alpha^\pm}\right)}$$

The in- and out- states can now be expressed as

$$\Psi_\alpha^\pm = \Phi_\alpha + \int d\beta \frac{T_{\beta\alpha}^\pm \Phi_\beta}{E_\alpha - E_\beta \pm i\epsilon}$$

and the S-matrix elements

$$\begin{aligned}
S_{\beta\alpha} &= (\Psi_\beta^-, \Psi_\alpha^+) \\
&= \left(\Phi_\beta + \int d\gamma \frac{T_{\gamma\beta}^-\Phi_\gamma}{E_\beta - E_\gamma - i\epsilon}, \Phi_\alpha + \int d\sigma \frac{T_{\sigma\alpha}^+\Phi_\sigma}{E_\alpha - E_\sigma + i\epsilon}\right) \\
&= \delta(\beta-\alpha) + \left(\Phi_\beta, \int d\sigma \frac{T_{\sigma\alpha}^+\Phi_\sigma}{E_\alpha - E_\sigma + i\epsilon}\right) + \left(\int d\gamma \frac{T_{\gamma\beta}^-\Phi_\gamma}{E_\beta - E_\gamma - i\epsilon}, \Phi_\alpha\right) \\
&\quad + \left(\int d\gamma \frac{T_{\gamma\beta}^-\Phi_\gamma}{E_\beta - E_\gamma - i\epsilon}, \int d\sigma \frac{T_{\sigma\alpha}^+\Phi_\sigma}{E_\alpha - E_\sigma + i\epsilon}\right) \\
&= \delta(\beta-\alpha) + \frac{T_{\beta\alpha}^+}{E_\alpha - E_\beta + i\epsilon} + \frac{T_{\alpha\beta}^{-*}}{E_\beta - E_\alpha + i\epsilon} + \int d\gamma \frac{T_{\gamma\beta}^{-*}}{E_\beta - E_\gamma + i\epsilon}\frac{T_{\gamma\alpha}^+}{E_\alpha - E_\gamma + i\epsilon}
\end{aligned}$$

that is

$$\boxed{S_{\beta\alpha} = \delta(\beta-\alpha) + \frac{T_{\beta\alpha}^+ - T_{\alpha\beta}^{-*}}{E_\alpha - E_\beta + i\epsilon} + \int d\gamma \frac{T_{\gamma\beta}^{-*}}{E_\beta - E_\gamma + i\epsilon}\frac{T_{\gamma\alpha}^+}{E_\alpha - E_\gamma + i\epsilon}}$$

with

$$T_{\beta\alpha}^+ - T_{\alpha\beta}^{-*} = \frac{g^2 u_\beta u_\alpha^*}{E_\alpha - E_\beta + i\epsilon}\left(\frac{1}{1-k_\alpha^+} - \frac{1}{1-k_\beta^+}\right)$$

Exercise 3.2

Suppose that a resonance of spin one is discovered in e^+-e^- scattering at total energy of 150 GeV and with a cross-section (in the center-of-mass frame, averaged over initial spins, and summed over final spins) for elastic e^+-e^- scattering at the peak of the resonance equal to 10^{-34} cm^2. What is the branching ratio for the decay of the resonant state R by the mode $R \to e^- + e^+$? What is the total cross-section for e^+-e^- scattering at the peak of the resonance ? (In answering both questions, ignore the non-resonant background scattering).

We're analizing the process

$$e^+ + e^- \to R \ \to \ e^+ + e^-$$

where R is a resonance. The resonance is at an energy $E = 150$ GeV and the cross section averaged over initial spins and summed over final spins of

$$\sigma = 10^{-34} \text{ cm}^2.$$

It follows from eq. (3.8.18) on page 163, that for a resonant scattering $ee \to ee$

$$\sigma(ee \to ee, E) = \frac{\pi(2j_R+1)}{k^2(2s_1+1)(2s_2+1)}\frac{\Gamma_{ee}\Gamma_{ee}}{(E-E_R)^2 + \Gamma^2/4}$$

at $E = E_R = 150$ GeV, we have, with $j_R = 1$ and $s_1 = s_2 = \frac{1}{2}$

$$\frac{3\pi}{4k^2}4\left(\frac{\Gamma_{ee}}{\Gamma}\right)^2 = 10^{-34} \text{ cm}^2.$$

from which

$$\frac{\Gamma_{ee}}{\Gamma} = k\frac{1}{\sqrt{3\pi}}10^{-17} \text{ cm}.$$

with $\hbar k = \frac{1}{2}\frac{E}{c}$ we get $k = 3.8 \cdot 10^{17}$ m^{-1}. So we have, finally

$$\boxed{\frac{\Gamma_{ee}}{\Gamma} = 1.24 \cdot 10^{-2}.}$$

The total cross-section is then given by equation (3.8.16) on page 162

$$\sigma_{\text{total}}(ee, E) = \frac{\pi(2j_R + 1)}{k^2(2s_1 + 1)(2s_2 + 1)} \frac{\Gamma\Gamma_{ee}}{(E - E_R)^2 + \Gamma^2/4}$$

and at the peak of the resonance

$$\boxed{\sigma_{\text{total}}(ee, E_R) = \frac{3\pi}{k^2}\frac{\Gamma_{ee}}{\Gamma} = 8.09 \cdot 10^{-33}\ \text{cm}^2 = 8.09\ \text{nb}.}$$

with $1\ \text{b} = 10^{-28}$ m.

Exercise 3.3

Express the differential cross-section for two-body scattering in the laboratory frame, in which one of the two particles is initially at rest, in terms of kinematic variables and the matrix element $M_{\beta\alpha}$. (Derive the result directly, without using the results derived in this chapter for the differential cross-section in the center-of-mass frame.)

Exercise 3.4

Derive the perturbation expansion (3.5.8) directly from the expansion (3.5.3) of the old fashioned perturbation theory.

Exercise 3.5

We can define 'standing wave' states Ψ^0_α by a modified version of the Lippmann-Schwinger equation

$$\Psi^0_\alpha = \Phi_\alpha + \frac{\mathcal{P}}{E_\alpha - H_0} V \Psi^0_\alpha.$$

Show that the matrix $K_{\beta\alpha} \equiv \pi\delta(E_\beta - E_\alpha)(\Phi_\beta, V\Psi^0_\alpha)$ is Hermitian.

Show how to express the S-matrix in terms of the K-matrix.

Exercise 3.6

Express the differential cross-section for elastic π^+-proton and π^--proton scattering in terms of the phase-shifts for states of definite total angular momentum, parity, and isospin.

Exercise 3.7

Show that the states $\Phi_{E,\mathbf{p}j\sigma sn}$ defined by equation (3.7.5) are correctly normalized to have the scalar product (3.7.6).

Chapitre 4

The Cluster Decomposition Principle

Exercise 4.1

Define the generating functionals for the S-matrix and its connected part

$$F[v] = 1 + \sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{1}{N!M!}\int v^*(q'_1)\dots v^*(q'_N)v(q_1)\dots v(q_M)S_{q'_1\dots q'_N q_1\dots q_M}dq'_1\dots dq'_N dq_1\dots dq_M$$

$$F^C[v] = \sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{1}{N!M!}\int v^*(q'_1)\dots v^*(q'_N)v(q_1)\dots v(q_M)S^C_{q'_1\dots q'_N q_1\dots q_M}dq'_1\dots dq'_N dq_1\dots dq_M$$

Derive a formula relating $F[v]$ and $F^C[v]$.(You may consider the purely bosonic case)

Starting from the first few-particles S-matrices, with stable particles, we have

$$S_{q'q} = S^C_{q'q}, \quad S_{q'_1q'_2q_1q_2} = S^C_{q'_1q'_2q_1q_2} + S^C_{q'_1q_1}S^C_{q'_2q_2} + S^C_{q'_1q_2}S^C_{q'_2q_1}$$

and

$$S_{q'_1q'_2q'_3q_1q_2q_3} = S^C_{q'_1q'_2q'_3q_1q_2q_3} + S^C_{q'_1q'_2q_1q_2}S^C_{q'_3q_3} + 8\text{ terms} + S^C_{q'_1q_1}S^C_{q'_2q_2}S^C_{q'_3q_3} + 5\text{ terms}$$

where the added terms are obteined by permutations of the indices of the previous term. For the 4-particles S-matri we have

$$\begin{aligned}S_{q'_1q'_2q'_3q'_4q_1q_2q_3q_4} = S^C_{q'_1q'_2q'_3q'_4q_1q_2q_3q_4} &+ S^C_{q'_1q_2q_1q_2}S^C_{q'_3q'_4q_3q_4} + 17\text{ terms} + S^C_{q'_1q'_2q'_3q_1q_2q_3}S^C_{q'_4q_4} + 15\text{ terms}\\ &+ S^C_{q'_1q'_2q_1q_2}S^C_{q'_3q_3}S^C_{q'_4q_4} + 71\text{ terms} + S^C_{q'_1q_1}S^C_{q'_2q_2}S^C_{q'_3q_3}S^C_{q'_4q_4} + 23\text{ terms}\end{aligned}$$

To ease the notation let us define

$$S_{11} = \int v^*(q')S_{q'q}v(q)dq'dq, \qquad S^C_{11} = \int v^*(q')S^C_{q'q}v(q)dq'dq$$

and so on. We have then

$$F[v] = 1 + S_{11} + \tfrac{1}{4}S_{22} + \tfrac{1}{36}S_{33} + \tfrac{1}{24^2}S_{44} + \dots$$

and

$$F^C[v] = S^C_{11} + \tfrac{1}{4}S^C_{22} + \tfrac{1}{36}S^C_{33} + \tfrac{1}{24^2}S^C_{44} + \dots$$

By using the relation between the first S and S^C, we can write

$$F[v] = 1+S^C_{11}+\tfrac{1}{4}\left(S^C_{22}+2{S^C_{11}}^2\right)+\tfrac{1}{36}\left(S^C_{33}+9S^C_{22}S^C_{11}+6{S^C_{11}}^3\right)+\tfrac{1}{24^2}\left(S^C_{44}+18{S^C_{22}}^2+16S^C_{33}S^C_{11}+72S^C_{22}{S^C_{11}}^2+24{S^C_{11}}^4\right)+\dots$$

We can then rearrange the terms as

$$\begin{aligned}F[v] = 1 &+ \left(S^C_{11} + \tfrac{1}{4}S^C_{22} + \tfrac{1}{36}S^C_{33} + \tfrac{1}{24^2}S^C_{44} + \dots\right) + \tfrac{1}{2}\left({S^C_{11}}^2 + \tfrac{1}{16}{S^C_{22}}^2 + \tfrac{1}{2}S^C_{22}S^C_{11} + \tfrac{1}{18}S^C_{33}S^C_{11} + \dots\right)\\ &+ \tfrac{1}{6}\left({S^C_{11}}^3 + \dots\right) + \tfrac{1}{24}\left(24{S^C_{11}}^4 + \dots\right) + \dots\end{aligned}$$

We recognize that this is the development of

$$F[v] = 1 + \left(S_{11}^C + \tfrac{1}{4}S_{22}^C + \tfrac{1}{36}S_{33}^C + \ldots\right) + \tfrac{1}{2!}\left(S_{11}^C + \tfrac{1}{4}S_{22}^C + \tfrac{1}{36}S_{33}^C + \ldots\right)^2 + \tfrac{1}{3!}\left(S_{11}^C + \tfrac{1}{4}S_{22}^C + \tfrac{1}{36}S_{33}^C + \ldots\right)^3 + \ldots$$

which is the exponential of $F^C[v]$

$$F[v] = \exp\left(F^C[v]\right).$$

We can also express this as

$$F^C[v] = \ln\left(F[v]\right).$$

Exercise 4.2

Consider an interaction

$$V = g\int d^3p_1 d^3p_2 d^3p_3 d^3p_4 \delta(p_1 + p_2 - p_3 - p_4) a^\dagger(p_1) a^\dagger(p_2) a(p_3) a(p_4),$$

where g is a real constant and $a(p)$ is the annihilation operator of a spinless boson of mass $M > 0$. Use perturbation theory to calculate the S-matrix element for scattering of these particles in the center-of-mass frame to order g^2. What is the corresponding differential cross-section ?

Exercise 4.3

A coherent state $\Phi_{q^n}\lambda$ is defined to be an eigenstate of the annihilation operators $a(q)$ with eigenvalues $\lambda(q)$. Construct such a statte as a superposition of multi-particles states $\Phi_{q_1 q_2 \ldots q_N}$.

A coherent state is an eigenstate of the annihilation operator $a = a(q)$

$$a\Phi_\lambda = \lambda\Phi_\lambda.$$

A multiparticle state on which is operated the annihilation operator a, should give a state with one particle state less. Since an eigenstate is invariant, that means that it must be a superposition of an infinite number of state of any number of particle

$$\Phi_\lambda = \sum_{k=0}^{\infty} f(k)(a^\dagger)^k \Phi_0$$

the state Φ_0 being the vacuum state. The state $(a^\dagger)^n\Phi_0 = \Phi_{q^n}$ is the state, prepared from the vacuum state with the creation operators $a^\dagger = a^\dagger(q)$ which create a particle in state q, and which contains then n particles in that state.

The coefficients $f(k)$ must now be chosen so that we recover an eigenstate of the annihilation operator with eigenvalue λ. To find f we operate on Φ_λ with a

$$a\Phi_\lambda = \sum_{k=0}^{\infty} f(k) a (a^\dagger)^k \Phi_0$$

and use the commutation relation

$$[a, a^\dagger] = 1$$

to move a to the right until it acts on Φ_0 to give 0. It is easy to show that

$$a(a^\dagger)^k = (a^\dagger)^k a + k(a^\dagger)^{k-1}$$

so

$$a\Phi_\lambda = \sum_{k=0}^{\infty} k f(k)(a^\dagger)^{k-1}\Phi_0 = \lambda \sum_{k=0}^{\infty} f(k)(a^\dagger)^k \Phi_0$$

This means that we must have

$$(k+1)f(k+1) = \lambda f(k)$$

From which, starting with $f(0) = c$

$$f(k+1) = \frac{\lambda}{k+1} f(k) = c\frac{\lambda^{k+1}}{(k+1)!}$$

So

$$\Phi_\lambda = c\sum_{k=0}^{\infty} \frac{(\lambda a^\dagger)^k}{k!} \Phi_0 = ce^{\lambda a^\dagger} \Phi_0.$$

The coefficient c can then be chosen so that the eigenstate is normalized.

Chapitre 5

Quantum Fields and Antiparticles

Exercise 5.1

Show that if the zero-momentum coefficient functions satisfy the conditions (5.1.23) and (5.1.24), then the coefficient functions (5.1.21) and (5.1.22) for arbitrary momentum satisfy the defining conditions Eqs. (5.1.19) and (5.1.20).

Exercise 5.2

Consider a free field $\psi_\ell^\mu(x)$ which annihilates and creates a self-charge.conjugate particle of spin $\frac{3}{2}$ and mass $m \neq 0$. Show how to calculate the coefficient functions $u_\ell^\mu(\mathbf{p}, \sigma)$, which multiply the annihilation operator $a(\mathbf{p}, \sigma)$ in this field, in such a way that the field transforms under Lorentz transformations like a Dirac field ψ_ℓ with an extra four-vector index μ. What field equations and algebraic and reality conditions does this field satisfy ? Evaluate the matrix $P^{\mu\nu}(p)$, defined (for $p^2 = -m^2$) by

$$\sum_\sigma u_\ell^\mu(\mathbf{p}, \sigma) u_m^{\nu *}(\mathbf{p}, \sigma) \equiv (2p^0)^{-1} P_{\ell m}^{\mu\nu}(p).$$

What are the commutation relations of this field ? How does the field transform under the inversions P, C, T ?

Exercise 5.3

Consider a free field $h^{\mu\nu}(x)$ satisfying $h^{\mu\nu}(x) = h^{\nu\mu}(x)$ and $h^\mu_\mu(x) = 0$, which annihilates and creates a particle of spin two and mass $m \neq 0$. Show how to calculate the coefficient functions $u^{\mu\nu}(\mathbf{p}, \sigma)$, which multiply the annihilation operators $a(\mathbf{p}, \sigma)$ in this field, in such a way that the field transforms under Lorentz transformations like a tensor. What field equations does this field satisfy ? Evaluate the function $P^{\mu\nu,\kappa\lambda}(p)$, defined by

$$\sum_\sigma u^{\mu\nu}(\mathbf{p}, \sigma) u^{\kappa\lambda *}(\mathbf{p}, \sigma) \equiv (2p^0)^{-1} P^{\mu\nu,\kappa\lambda}(p).$$

What are the commutation relations of this field ? How does the field transform under the inversions P, C, T ?

Exercise 5.4

Show that the fields for a massless particle of spin j of type $(A, A + j)$ or $(B + j, B)$ are the $2A$th or $2B$th derivatives of fields of type $(0, j)$ or $(j, 0)$ respectively.

Exercise 5.5

Work out the transformation properties of fields of transformation type $(j, 0) + (0, j)$ for massless particles of helicity $\pm j$ under the inversions P, C, T.

Exercise 5.6

Consider a generalized Dirac field ψ that transforms according to the $(j, 0) + (0, j)$ representation of the homogeneous Lorentz group. List the tensors that can be formed from products of the components of ψ and $\psi^\dagger$. Check your results against what we found for $j = \frac{1}{2}$.

Exercise 5.7

Consider a general field ψ_{ab} describing particles of spin j and mass $m \neq 0$, that transforms according to the (A, B) representation of the homogeneous Lorentz group. Suppose it has an interaction Hamiltonian of the form

$$V = \int d^3x \left[\psi_{ab}(x) J^{ab}(x) + J^{ab\dagger}(x) \psi^\dagger_{ab}(x)\right],$$

where J^{ab} is an external c-number current. What is the asymptotic behavior of the matrix element for emitting these particles for energy $E \gg m$ and definite helicity ? (Assume that the Fourier transform of the current has values for different a, b that are of the same order of magnitude, and that do not depend strongly on E.)

Chapitre 6

The Feynman Rules

Exercise 6.1

Consider the theory of a real scalar field ϕ, with interaction (in the interaction picture) $V = g\int d^3x\phi(x)^3/3!$. Calculate the connected S-matrix element for scalar-scalar scattering to second order in g, doing all integrals. Use the results to calculate the differential cross-section for scalar-scalar scattering in the center-of-mass system.

Exercise 6.2

Exercise 6.3

Exercise 6.4

Exercise 6.5

Chapitre 7

The Canonical Formalism

Septième partie

S. Weinberg II. Modern Applications

[Wei96]

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